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Robust output regulation of variable structure systems with multivalued controls

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***Abstract*— We consider the problem of robust output regulation for a class of passive linear systems. We take a 'control by interconnection of systems' approach, where the controller is defined by means of a multivalued function. The resulting closed-loop system can be cast into the form of a variable structure system with interesting properties: output feedback (perfect knowledge of the plant state is not required) and perfect regulation despite parametric uncertainty and *unmatched* disturbances. The methodology is illustrated through a physical and an abstract example.**

I. Introduction

The classical framework for studying systems with sliding motions is Filippov's theory of differential equations with discontinuous right-hand sides [1]. For the purposes of analysis, the usual signum function is replaced by a multivalued function. This results in a differential inclusion in place of the original differential equation. In this paper we explore the possibility of constructing controllers using multivalued functions other than the well-known multivalued signum.

While the literature on the *analysis* of non-smooth systems involving multivalued functions is extensive (e.g. [1], [2], [3], [4], [5]), the problem of *designing* a control system using general multivalued laws has been substantially less explored. A notable contribution in this direction takes place in viability theory [6, Ch. 11].

For concreteness, we focus on the robust output regulation problem for passive systems using multivalued passive controllers. This approach takes us to a complementarity system [7] that can be equivalently written as a variable structure system. The proposed controller:

- Achieves insensitivity of the regulated output vis-à-vis *unmatched* parametric uncertainty and external disturbances;
- Does not require knowledge of the full state.

Note that these properties stand in sharp contrast to those of classical sliding-mode controllers, which are robust vis-à-vis matched uncertainty only and typically require knowledge of the full state.

The paper is organized as follows. The main assumptions and controller structure are introduced in Section II. In Section III we show asymptotic stability and finite-time convergence of the output. Section IV discusses some of the practical issues regarding the implementation of the proposed multivalued control law. We provide some concluding remarks in Section V.

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II. Control with Multivalued Feedback Laws

In this section we define the class of systems that we wish to control (affine and strictly passive) and the class of controllers that we will focus on (multivalued and passive as well).

A. The Plant

Let Σ_1 be an affine uncertain system of the form

$$\dot{x} = Ax + B_{\mathrm{u}} u_1 + B_{\mathrm{v}} v \tag{1a}$$

$$y_1 = Cx + Du_1 \; , \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state at time t (we omit time arguments for brevity), $u_1, y_1 \in \mathbb{R}$ are *port* variables. Matrices A, B_{u}, B_{v}, C and D are constant and of suitable dimensions. The term $v \in \mathbb{R}$ accounts for an uncertain exogenous input. In order to convey the main ideas and to avoid loosing the reader on technicalities, we restrict our attention to the case were v is constant. However, it is worth mentioning that, at the expense of more complicated proofs and formulae, it is possible to extend the class of exogenous inputs to one that includes bounded time-varying functions, so that more realistic perturbations can be taken into account (cf. the example on Section IV-A).

Our problem consists in regulating y_1 to a desired fixed value y_{d}, despite the system uncertainty. For concreteness, we assume $y_{\mathrm{d}} > 0$. This is without loss of generality, since the following analysis can be easily adapted to the case $y_{\mathrm{d}} \leq 0$.

Notice that, for $D = 0$ and $B_{\mathrm{u}} = B_{\mathrm{v}}$, the problem reduces to a standard sliding-mode control problem with matched disturbances. We depart from these standard assumptions and make the following instead.

Assumption 1: There exists a (possibly unknown) matrix $P = P^\top > 0$ such that

$$\begin{bmatrix} PA + A^\top P & PB_{\mathrm{u}} - C^\top \\ B_{\mathrm{u}}^\top P - C & -2D \end{bmatrix} < 0 \; . \tag{2}$$

Roughly speaking, inequality (2) requires the unforced system

$$\dot{x} = Ax + B_{\mathrm{u}} u_1 \tag{3a}$$

$$y_1 = Cx + Du_1 \tag{3b}$$

to be strictly passive (see [8, p. 34] and the proof of Lemma 1 below). Notice, however, that precise knowledge of the storage function

$$V(x) = \frac{1}{2} x^\top P x \tag{4}$$

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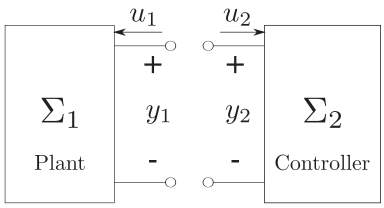

Fig. 1. Interconnection of a controller to a plant.

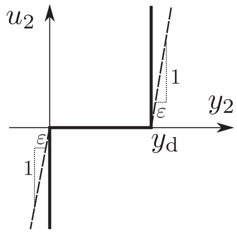

Fig. 2. Graph of the multifunction $U(y_2) = \partial\Psi_\mathcal{S}(y_2)$ (solid line) and a continuous approximation (dotted).

is not required, so the design methodology we present later has tolerance for large uncertainty, both in the system parameters and the external, unmatched, perturbation (albeit constant, at present). In many cases, inequality (2) can be established on physical grounds, without the need to verify (3) explicitly (see the example on Section III-A).

The following lemma establishes the precise physical meaning of (2), together with a technical result needed later for proving the main theorem.

Lemma 1: The following statements are equivalent:

i) The unforced system (3) is strictly passive with storage function (4), i.e., $\dot{V}(x) < u \cdot y$ for all x, u such that $\|x\| + \|u\| \neq 0$.
ii) Inequality (2) holds.
iii) The inequality[1]

$$\begin{bmatrix} P\tilde{A} + \tilde{A}^\top P & PB_\mathrm{u} + C^\top \\ B_\mathrm{u}^\top P + C & -2D \end{bmatrix} < 0 \tag{5}$$

holds with

$$\tilde{A} := A - B_\mathrm{u} D^{-1} C \ . \tag{6}$$

See the appendix for a proof.

B. Ideal Multivalued Controls

Let Y be a metric space and 2^Y be the class of all the subsets of Y. Recall that a multivalued function F is a map from a metric space X to 2^Y.

Definition 1: The *graph* of a multivalued function F is the subset of the product space $X \times Y$ defined by $\mathrm{Graph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$.

Now, let $u_2, y_2 \in \mathbb{R}$ be the controller port variables. The behavior of the ideal controller is characterized by a multi-valued law $U : \mathbb{R} \to 2^\mathbb{R}$. Allow us to defer the mandatory discussion on the implementation of such a controller until Section IV, and allow us to state the following assumption instead.

Assumption 2: The controller is static and passive, i.e., $u_2 \cdot y_2 \geq 0$ for all $(u_2, y_2) \in \mathrm{Graph}(U)$.

The plant and controller are interconnected using the interconnection pattern

$$y_1 = y_2 \quad \text{and} \quad u_1 = -u_2 \ , \tag{7}$$

as in Fig. 1. Note that the interconnection is power-preserving: $u_1 \cdot y_1 + u_2 \cdot y_2 = 0$ (the interested reader is referred to, e.g., [10]). The overall interconnected system is thus

$$\dot{x} = Ax - B_\mathrm{u} u_2 + B_\mathrm{d} v \tag{8a}$$
$$y_2 = Cx - Du_2 \tag{8b}$$
$$u_2 \in U(y_2) \tag{8c}$$

and our goal is to construct $U(y_2)$ such that $y_2 \to y_\mathrm{d}$.

For the sake of argument note that, in the absence of Assumption 2, the multivalued function

$$U(y_2) = \begin{cases} \mathbb{R} & \text{if } y_2 = y_\mathrm{d} \\ \emptyset & \text{if } y_2 \neq y_\mathrm{d} \end{cases} \tag{9}$$

would solve our regulator problem. Indeed, the solution of (8) with (9) would be simply $u_2 = D^{-1}(Cx - y_\mathrm{d})$, which ensures that $y_2 = y_\mathrm{d}$. Unfortunately, such controller is not passive. Furthermore, D and Cx might be unavailable or available with large uncertainty. It seems however reasonable to 'passivize' (9) in the following way: Define the set $\mathcal{S} = [0, y_\mathrm{d}]$ and write its indicator function

$$\Psi_\mathcal{S}(z) = \begin{cases} 0 & \text{if } z \in \mathcal{S} \\ +\infty & \text{if } z \notin \mathcal{S} \end{cases} .$$

We can now write the proposed multivalued control law as

$$U(y_2) = \partial\Psi_\mathcal{S}(y_2) \ , \tag{10}$$

where $\partial\Psi(z)$ denotes the subdifferential of $\Psi_\mathcal{S}$ at z, as defined in classical convex analysis [3, p. 28]. The graph of (10) is depicted in Fig. 2, which shows that the controller is indeed passive (the graph belongs to the first and third quadrants only). The graph is that of the characteristic of an ideal Zener diode with breakdown voltage y_d (see [2] for more details on modelling and analysis of circuits with Zener diodes).

III. Stability

Let us analyse the stability properties of the controlled system. From (8b), (8c) and (10), we know that $Cx - y_2 \in D \cdot \partial\Psi_\mathcal{S}(y_2)$. Since D is a positive scalar (cf. (2)) and $\partial\Psi_\mathcal{S}(y)$ is a cone [3, p. 35], D can be absorbed into $\partial\Psi_\mathcal{S}(y)$, so that

$$Cx - y_2 \in \partial\Psi_\mathcal{S}(y_2) \ .$$

It follows from classical convex analysis (see, e.g., [3, p. 38]) that

$$y_2 = \mathrm{Proj}_\mathcal{S}(Cx) \ , \tag{11}$$

[1] See [9] for an appearance of this LMI in an optimal control context.

where $\mathrm{Proj}_{\mathcal{S}}(z)$ is the projection of a point z onto $\mathcal{S}$, i.e., $\mathrm{Proj}_{\mathcal{S}}(z) = \arg\min_{\bar{z}\in\mathcal{S}} \|\bar{z} - z\|$.

Remark 1: The projection operation (11) reflects the controller's robust regulation property. The exact value of Cx is unimportant: As long as $Cx \geq y_{\mathrm{d}}$, we will have $y_2 = y_{\mathrm{d}}$.

Substituting (11) into (8b) gives

$$u_2 = D^{-1}\left[Cx - \mathrm{Proj}_{\mathcal{S}}(Cx)\right]$$

which, when substituted in (8a), gives in turn

$$\dot{x} = \tilde{A}x + B_{\mathrm{u}}D^{-1}\,\mathrm{Proj}_{\mathcal{S}}(Cx) + B_{\mathrm{v}}v \tag{12}$$

with $\tilde{A}$ as in (6). Recall that the projection is a Lipschitz continuous operator [11, p. 118], so (12) is an ordinary differential equation with a unique solution in Carathéodory's sense.

System (12) is a *variable structure system* that can be decomposed into three affine systems S_1, S_2 and S_3:

$$S_1 : \dot{x} = \tilde{A}x + B_{\mathrm{v}}v\,, \quad x \in \mathcal{H}_1\,, \tag{13a}$$
$$S_2 : \dot{x} = Ax + B_{\mathrm{v}}v\,, \quad x \in \bar{\mathcal{H}}_1 \cap \mathcal{H}_2\,, \tag{13b}$$
$$S_3 : \dot{x} = \tilde{A}x + B_{\mathrm{u}}D^{-1}y_{\mathrm{d}} + B_{\mathrm{v}}v\,, \quad x \in \bar{\mathcal{H}}_2\,, \tag{13c}$$

where $\mathcal{H}_1$, $\mathcal{H}_2$ are open half-spaces given by

$$\mathcal{H}_1 = \{x \in \mathbb{R}^n : Cx < 0\}\,, \quad \mathcal{H}_2 = \{x \in \mathbb{R}^n : Cx < y_d\}\,,$$

and $\bar{\mathcal{H}}_1$, $\bar{\mathcal{H}}_2$ are their complements in $\mathbb{R}^n$.

For $y_{\mathrm{d}} > 0$ we have the relation $\mathcal{H}_1 \subset \mathcal{H}_2$ and hence $\bar{\mathcal{H}}_2 \subset \bar{\mathcal{H}}_1$. It follows from Remark 1 that, for output regulation, one must drive the system state into $\bar{\mathcal{H}}_2$ and maintain it there. A necessary condition for output regulation is thus that S_3 has its stable equilibrium

$$\bar{x}_{(3)} := -\tilde{A}^{-1}(B_{\mathrm{u}}D^{-1}y_{\mathrm{d}} + B_{\mathrm{v}}v) \tag{14}$$

inside its own domain of definition, $\bar{\mathcal{H}}_2$. The following theorem states that this condition is sufficient as well.

Theorem 1 (Main result): Consider the closed-loop system (8). Suppose that Assumption 1 holds and that

$$\bar{x}_{(3)} \in \bar{\mathcal{H}}_2\,, \tag{15}$$

i.e., $C\bar{x}_{(3)} \geq y_{\mathrm{d}}$ with $\bar{x}_{(3)}$ as in (14). Then, $x = \bar{x}_{(3)}$ is a globally asymptotically stable equilibrium. Furthermore, y_2 converges to y_{d} in finite time.

Proof: In order to prove the statement, consider the displacement $e := x - \bar{x}_{(3)}$ and the candidate Lyapunov function $V(e) = \frac{1}{2}e^\top Pe$ with $P = P^\top > 0$ satisfying (2). It has already been established that (8) is equivalent to a variable structure system, so let us compute the derivative of $V(e)$ along the trajectories given by (13) and show that is negative definite. There are three cases.

For (13a) we have the dynamics $\dot{e} = \tilde{A}e - B_{\mathrm{u}}D^{-1}y_{\mathrm{d}}$ and the constraint $e + \bar{x}_{(3)} \in \mathcal{H}_1$. The time-derivative of $V(e)$ is

$$\dot{V}(e) = -\frac{1}{2}e^\top\tilde{Q}e - e^\top PB_{\mathrm{u}}D^{-1}y_{\mathrm{d}}$$

with $\tilde{Q} := -(P\tilde{A} + \tilde{A}P)$.

We will show that $\dot{V}(e) < 0$ by proving that the minimum of

$$f_1(e) = \frac{1}{2}e^\top\tilde{Q}e + e^\top PB_{\mathrm{u}}D^{-1}y_{\mathrm{d}}\,,$$

subject to $g_1(e) = Ce + y_d \leq 0$, is positive. The constraint $g_1(e) \leq 0$ is a simple consequence of the inclusions $\bar{x}_{(3)} \in \bar{\mathcal{H}}_2$ and $e + \bar{x}_{(3)} \in \mathcal{H}_1$. The optimization problem is quadratic and strictly convex in f_1, and convex in g_1, so the usual first-order necessary conditions for optimality are also sufficient. Using Karush-Kuhn-Tucker (KKT) conditions (see [12, p. 309]), the constrained minimum $e^*_{(1)}$ must satisfy

$$\nabla f_1(e^*_{(1)}) + \mu\nabla g_1(e^*_{(1)}) = 0$$
$$\mu \geq 0\,, \quad \mu \cdot g_1(e^*_{(1)}) = 0\,.$$

The first condition translates to

$$e^*_{(1)} = -\tilde{Q}^{-1}\left(PB_{\mathrm{u}}D^{-1}y_d + \mu C^\top\right)\,.$$

Suppose that the constraints are inactive ($\mu = 0$). Then, evaluation of g_1 in $e^*_{(1)}$ leads to

$$g_1(e^*_{(1)}) = \left(D - C\tilde{Q}^{-1}PB_{\mathrm{u}}\right)D^{-1}y_{\mathrm{d}}\,.$$

By taking Schur complements, we know that (5) implies

$$2\left(D - C\tilde{Q}^{-1}PB_{\mathrm{u}}\right) - B_{\mathrm{u}}^\top P\tilde{Q}^{-1}PB_{\mathrm{u}} - C\tilde{Q}^{-1}C^\top > 0\,,$$

which in turn implies $D - C\tilde{Q}^{-1}PB_{\mathrm{u}} > 0$ and in consequence that the constraint $g_1(e^*) \leq 0$ is violated. Hence, $\mu > 0$ and the constraint is satisfied with equality, i.e. $y_{\mathrm{d}} = -Ce^*_{(1)}$. Substituting this expression in f_1 gives the upper bound

$$\dot{V}(e) = -f(e) \leq -\frac{1}{2}e^{*\top}_{(1)}\tilde{Q}e^*_{(1)} + e^{*\top}_{(1)}PB_{\mathrm{u}}D^{-1}Ce^*_{(1)}\,.$$

Setting $Q := -(PA + AP)$ gives $\dot{V}(e) \leq -\frac{1}{2}e^{*\top}_{(1)}Qe^*_{(1)}$. Since $Q > 0$ by Assumption 1, we conclude that $\dot{V}(e)$ is negative definite for all $e + \bar{x}_{(3)} \in \mathcal{H}_1$.

For (13b) we have the dynamics

$$\dot{e} = Ae + B_{\mathrm{u}}D^{-1}\left(C\bar{x}_{(3)} - y_{\mathrm{d}}\right)$$

and the constraint $e + \bar{x}_{(3)} \in \bar{\mathcal{H}}_1 \cap \mathcal{H}_2$. The time-derivative of $V(e)$ is

$$\dot{V}(e) = -\frac{1}{2}e^\top Qe + e^\top PB_{\mathrm{u}}D^{-1}\left(C\bar{x}_{(3)} - y_{\mathrm{d}}\right)\,.$$

Similarly to the previous case, we seek to find the minimum of

$$f_2(e) = \frac{1}{2}e^\top Qe - e^\top PB_{\mathrm{u}}D^{-1}\left(C\bar{x}_{(3)} - y_d\right)$$

subject to $g_2(e) = C(e + \bar{x}_{(3)}) - y_d \leq 0$. The first KKT condition gives

$$e^*_{(2)} = Q^{-1}\left(PB_{\mathrm{u}}D^{-1}\left(C\bar{x}_{(3)} - y_d\right) - \mu C^\top\right)$$

Suppose again that $\mu = 0$. Then, evaluation of g_2 at $e^*_{(2)}$ gives

$$g_2(e^*_{(2)}) = \left(D + CQ^{-1}PB_{\mathrm{u}}\right)D^{-1}\left(C\bar{x}_{(3)} - y_{\mathrm{d}}\right)\,.$$

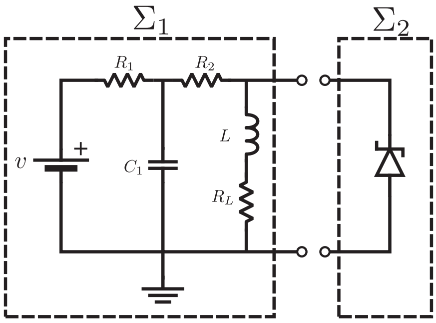

Fig. 3. An RLC circuit with a source. The goal is to regulate the voltage at the port of Σ_1.

Using the same Schur argument, we can show that (2) implies $D + CQ^{-1}PB_{\rm u} > 0$, which violates the constraint on account of $D > 0$ and the inclusion $\bar{x}_{(3)} \in \bar{\mathcal{H}}_2$. Thus, the constraint is again active and $C\bar{x}_{(3)} - y_{\rm d} = -Ce^*_{(2)}$. Substituting this condition in f_2 gives the upper bound

$$\begin{aligned}\dot{V}(e) &\leq -\frac{1}{2}e^{*\top}_{(2)}Qe^*_{(2)} - e^{*\top}_{(2)}PB_{\rm u}D^{-1}C_{(2)}e^* \\ &= -\frac{1}{2}e^{*\top}_{(2)}\tilde{Q}e^*_{(2)} < 0 \ .\end{aligned}$$

Therefore, $\dot{V}(e)$ is negative definite for $e + \bar{x}_{(3)} \in \bar{\mathcal{H}}_1 \cap \mathcal{H}_2$.

For (13c) the dynamics is governed by $\dot{e} = \tilde{A}e$ and the derivative of $V(e)$ is given by $\dot{V}(e) = -\frac{1}{2}e^\top\tilde{Q}e < 0$, $e \neq 0$. Therefore, $\dot{V}(e)$ is strictly negative for all non zero e such that $e + \bar{x}_{(3)} \in \bar{\mathcal{H}}_2$.

In summary, $\dot{V}(e)$ is negative definite in $\mathbb{R}^n$. The equilibrium $x = \bar{x}_{(3)}$ is thus globally asymptotically stable and, by Remark 1, y_2 converges to $y_{\rm d}$. To see why the convergence takes place in finite time, notice that, outside $\bar{\mathcal{H}}_2$, $\dot{V}(e)$ is bounded from above by the constant $-\frac{1}{2}\min\left\{e^{*\top}_{(1)}Qe^*_{(1)}, e^{*\top}_{(2)}\tilde{Q}e^*_{(2)}\right\}$.

■

A. Example, an RLC circuit

To illustrate the use of the theorem, we present an electrical circuit application. Consider the circuit described by the diagram shown in Fig. 3. We wish to regulate the voltage at the port of Σ_1 to a desired value $y_{\rm d}$.

We take the electric charge in the capacitor and the magnetic flux in the inductor as state variables. The port variables u_1 and y_1 are the current and the voltage, respectively. Applying Kirchoff's laws and following the current/voltage sign convention of Fig. 1, one obtains the state-variable representation

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{1}{C_1R_1} & -\frac{1}{L} \\ \frac{1}{C_1} & -\frac{R_2+R_L}{L}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}1 \\ R_2\end{bmatrix}u_1 + \begin{bmatrix}\frac{1}{R_1} \\ 0\end{bmatrix}v \tag{16a}$$

$$y_1 = \begin{bmatrix}\frac{1}{C_1} & -\frac{R_2}{L}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + R_2u_1 \ . \tag{16b}$$

Assumption 1 can be established on physical grounds: The unforced system is passive, since it results from the interconnection of passive elements. Let us verify it formally. A natural selection for P is the one for which $V(x)$ in (4) corresponds to the energy stored in the capacitor and the inductor, i.e.,

$$P = \begin{bmatrix}\frac{1}{C_1} & 0 \\ 0 & \frac{1}{L}\end{bmatrix} \ .$$

Replacing the system parameters in (2) leads to the matrix inequality

$$\begin{bmatrix}\frac{1}{C_1^2R_1} & 0 & 0 \\ 0 & \frac{R_2+R_L}{L^2} & -\frac{R_2}{L} \\ 0 & -\frac{R_2}{L} & R_2\end{bmatrix} > 0 \ ,$$

which can be readily verified by computing the determinants of the leading minors (note that it is not necessary to known the specific values of the system parameters).

The equilibrium (14) is given by

$$\bar{x}_{(3)} = \begin{bmatrix}\frac{C_1R_1R_2}{R_1+R_2}\left(\frac{v}{R_1} + \frac{y_{\rm d}}{R_2}\right) \\ \frac{L}{R_L}y_{\rm d}\end{bmatrix} \ .$$

To verify (15) we will check that $C\bar{x}_{(3)} - y_{\rm d} \geq 0$. Substitution of the system parameters leads to the following sufficient condition for output regulation:

$$v\frac{R_L}{R_1 + R_2 + R_L} \geq y_{\rm d} \ . \tag{17}$$

This condition has a physical interpretation: In steady state, the capacitor and the inductor can be replaced by an open and a short circuit, respectively. The voltage that results from the voltage divider given by R_1, R_2 and R_L is greater or equal to the desired one. It is clear that, for a passive controller, the condition is also necessary, so (15) is a *tight condition*.

It can be readily verified that (17) is fulfilled with the system parameters given in Table I and the desired output voltage $y_{\rm d} = 3\,[V]$. Figure 4 shows the phase plane that

R_1	R_2	R_L	C_1	L	v
$1\,[\Omega]$	$2\,[\Omega]$	$1\,[\Omega]$	$1\,[F]$	$1\,[H]$	$15\,[V]$

TABLE I

SYSTEM PARAMETERS FOR THE CIRCUIT DEPICTED IN FIG. 3.

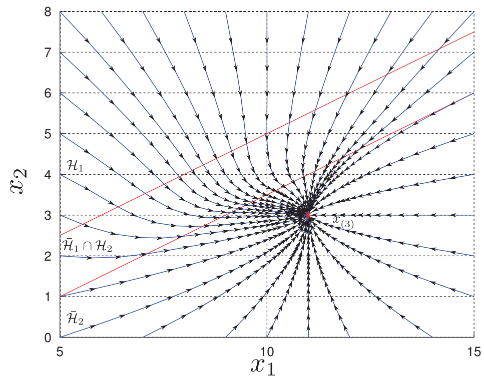

Fig. 4. Phase plane of (16) with the parameters given in Table I. Regulation is achieved with the controller $u_2 \in \partial\Psi_{\mathcal{S}}(y_2)$.

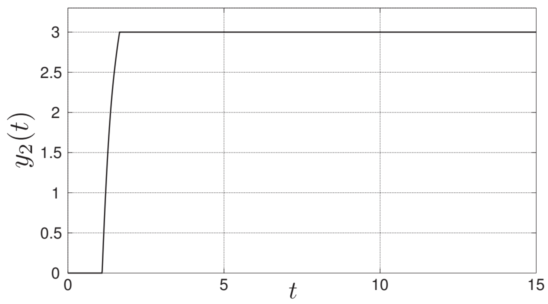

Fig. 5. Time history of y_2 for (16) with the parameters given in Table I. The system is regulated by the controller $u_2 \in \partial\Psi_{\mathcal{S}}(y_2)$. The finite-time convergence of y_2 to $y_{\rm d}$ is verified.

results from interconnecting (16) with the controller $u_2 \in \partial\Psi_{\mathcal{S}}(y_2)$ using the interconnection (7). Figure 5 shows the time history of y_2, where the convergence in finite time can be verified, despite the uncertainties in the system parameters.

B. Example, an abstract system

The controller $u_2 \in \partial\Psi_{\mathcal{S}}(y_2)$ is sufficiently general to be applied to a system which is not necessarily endowed with a physical structure. Consider an abstract system of the form (1) with

$$\begin{aligned} A &= \begin{bmatrix} -1.9395 & -0.1376 & 2.1477 \\ 0.0360 & -2.0806 & 1.2583 \\ -0.9549 & 0.0705 & -3.9808 \end{bmatrix}, \\ B_{\rm u} &= \begin{bmatrix} -0.5852 \\ 0.6481 \\ 1.1450 \end{bmatrix}, \quad B_{\rm v} = \begin{bmatrix} 0.121 \\ -1.9115 \\ -0.1259 \end{bmatrix}, \\ C &= \begin{bmatrix} 4.2544 & -2.1044 & 4.2387 \end{bmatrix}, \quad D = 5 \end{aligned} \tag{18}$$

and $v = 10$. Suppose we want to regulate the output to the set-point $y_{\rm d} = 5$.

Let us verify the assumptions of the main theorem. The equilibrium point of (13c) is

$$\bar{x}_{(3)} = \begin{bmatrix} 0.6199 & -10.9758 & -1.4890 \end{bmatrix}^\top$$

and it is in $\bar{\mathcal{H}}_2$ since $C\bar{x}_{(3)} - y_{\rm d} = 14.4231 > 0$. Using Sedumi to solve the LMI (2) we obtain

$$P = \begin{bmatrix} 6.1002 & -0.8413 & 1.2888 \\ -0.8413 & 5.9186 & 0.4392 \\ 1.2888 & 0.4392 & 3.6925 \end{bmatrix},$$

which is indeed symmetric and positive definite with eigenvalues in $\{2.9436, 5.7403, 7.0274\}$. All the assumptions of Theorem 1 are satisfied and the convergence of y_2 to $y_{\rm d}$ is assured by the control $u_2 \in \partial\Psi_{\mathcal{S}}(y_2)$. The time history of a trajectory and its corresponding output is shown in Figs. 6 and 7, respectively.

IV. Rational and Practical Considerations

At a first glance, the multivalued model $u_2 \in U(y_2)$ might seem unorthodox but, actually, multivalued control laws fit well within Jan Willems' behavioral framework [13], where

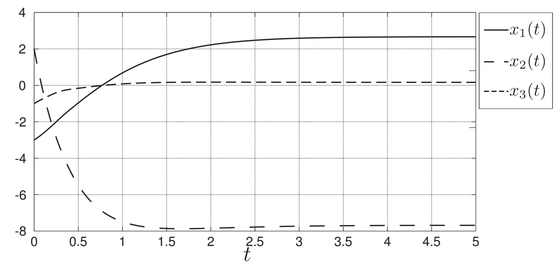

Fig. 6. State evolution of regulated plant (1) with parameters (18).

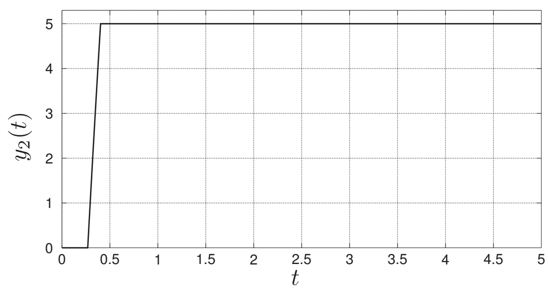

Fig. 7. Finite-time convergence of the output of regulated plant (1) with parameters (18).

the behavior of a system is expressed as a subset of the universe of all possible events (in our specific case, the controller is specified as $\mathrm{Graph}(U) \subset \mathbb{R}^2$).

In such framework, the notion of causality that comes embedded in the usual input–output setting is intentionally blurred. Traditionally, one tends to think of u_1 as a control and of y_1 as an output, so that u_1 plays the role of the cause and y_1 plays the role of the effect. However, for $D \neq 0$, the asymmetry between these variables is artificial and the causal structure relating u_1 and y_1 turns out to be misleading. To see this, one can simply solve for u_1 in (1b), substitute in (1a) and obtain an equivalent system with the roles of u_1 and y_1 reversed. When reversing the roles of u_1 and y_1, the multivalued function depicted in Fig. 2 results in the familiar signum function with a bias.

The multivalued model can also be understood as a high-level description of a controller, where the actual implementation depends on the particular application. We can think of (at least) three ways to implement the controller:

i) At every sampling time, the inclusion is solved (note that, for every x, there exists a unique solution of the inclusion $u_2 \in U(y_2)$). This method has the drawback of requiring perfect knowledge of Cx and D.
ii) The controller is a *physical* device (i.e., as opposed to a signal processor implemented on a computer). An example is given for the RLC circuit, where the controller can be implemented with a Zener diode, the characteristic of which is equal to the desired multivalued function. See [14] for more details and examples of this control approach.
iii) The multivalued control law is regularized, i.e., it is replaced by a continuous single-valued approximation.

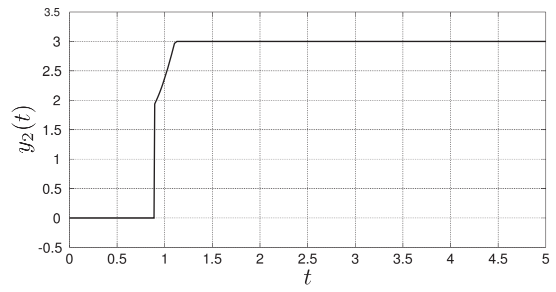

Fig. 8. Time history of y_2 for (16) with the parameters given in Table I, regulated by regularized controller (19) and subject to the perturbation $d(t) = 5\sin(t)$. The finite-time convergence of y_2 to y_d is still verified.

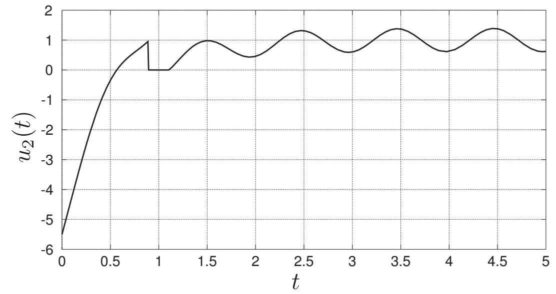

Fig. 9. Regularized control (19) for (16) with the parameters given in Table I and subject to the perturbation $d(t) = 5\sin(t)$.

A. Example, the RLC circuit subject to regularization and time-varying perturbations

In order to support claim iii) above, we approximate the multivalued control law (10) by a the single-valued function

$$u_2 = \begin{cases} \frac{1}{\varepsilon^2}(y_2 - \varepsilon) & \text{if } y_2 < \varepsilon \\ 0 & \text{if } \varepsilon \le y_2 \le y_d - \varepsilon \\ \frac{1}{\varepsilon^2}(y_2 - y_d + \varepsilon) & \text{if } y_d - \varepsilon < y_2 \end{cases}, \quad (19)$$

where $0 < \varepsilon << 1$ (see Fig. 2). We apply this control law to the circuit example described in Section III-A and take the opportunity to add a time-varying perturbation. To do so, we replace (1a) by $\dot{x} = Ax + B_u u_1 + B_v(v + d)$ with $d(t) = 5\sin(t)$. The system output and input are shown in Figs. 8 and 9, respectively. The output is perfectly regulated (within a tolerance determined by ε) in finite time while the control can be seen to compensate the perturbation exactly.

V. Conclusions and Future Works

The paper shows interesting features (not shared by traditional sliding-mode controllers) that a multivalued, nonsmooth controller can achieve in an output regulation context. Namely, robust output regulation (where robustness is against unmatched parametric and exogenous uncertainty) and output feedback (no need to measure the complete state).

The methodology presented can also be useful, e.g., when the relative degree of the regulated output is zero and the classical sliding mode control methodology cannot be used to design a robust control that regulates it.

The interpretation of our main assumption in terms of passivity opens an opportunity for extending the results to nonlinear passive systems.

Appendix

Proof: [of Lemma 1] Let us compute the derivative of (4) along the trajectories of (3a):

$$\begin{aligned}\dot{V}(x) &= \frac{1}{2}x^\top \left(PA + A^\top P\right)x + uB_\mathrm{u}^\top Px \\ &= -\frac{1}{2}x^\top Qx + u\cdot\left(B_\mathrm{u}^\top Px + y - Cx - Du\right),\end{aligned}$$

where the last equation follows by adding a null term obtained from (3b). Equivalently, we can write

$$\dot{V}(x) = \frac{1}{2}\begin{bmatrix} x^\top & u\end{bmatrix}\begin{bmatrix} PA + A^\top P & PB_\mathrm{u} - C^\top \\ B_\mathrm{u}^\top P - C & -2D\end{bmatrix}\begin{bmatrix} x \\ u\end{bmatrix} + u\cdot y,$$

which clearly establishes the equivalence between i) and ii).

Now, consider the linear invertible transformation

$$\begin{bmatrix} x \\ u\end{bmatrix} = \begin{bmatrix} I & 0 \\ -D^{-1}C & 1\end{bmatrix}\begin{bmatrix} \xi \\ \nu\end{bmatrix}.$$

It is straightforward to verify that

$$\begin{aligned}&\begin{bmatrix} x^\top & u\end{bmatrix}\begin{bmatrix} PA + A^\top P & PB_\mathrm{u} - C \\ B_\mathrm{u}^\top P - C & -2D\end{bmatrix}\begin{bmatrix} x \\ u\end{bmatrix} = \\ &\qquad \begin{bmatrix} \xi^\top & \nu\end{bmatrix}\begin{bmatrix} P\tilde{A} + \tilde{A}^\top P & PB_\mathrm{u} + C \\ B_\mathrm{u}^\top P + C & -2D\end{bmatrix}\begin{bmatrix} \xi \\ \nu\end{bmatrix},\end{aligned}$$

which proves the equivalence between ii) and iii). ■

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