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Multiplicity-induced dominance in stabilization of state predictors for time-delay systems

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Abstract: We propose a predictor tuning method for a class of uniformly observable nonlinear time-delay systems. This method is based on the multiplicity-induced-dominancy property of quasipolynomials. The predictor gains are chosen in such a way that a dominant root is located in the left half-plane, thus the prediction error converges exponentially to zero. For illustration purposes, a state prediction for the feedback control of a simple pendulum is designed and simulation results are presented.

Keywords: Time-delay nonlinear systems; observer-predictor design; multiplicity-induced-dominancy.

1. INTRODUCTION

Due to technical or economic constraints, it is usually impossible to directly implement full-state feedback control laws. Rather, the state is estimated using input–output information, and the true state is replaced by the estimated state. Reconstructing the state requires the plant to satisfy certain observability conditions. Since the control input is not known a priori, the stronger condition of uniform observability (observability in the presence of any input) is required (Besançon, 2007).

A time-delay h in the system input is well known to deteriorate the closed-loop behavior and possibly cause instability. The use of observer-predictors is a remedy for this issue. They estimate the future state at time $t + h$, so that the input delay is compensated when the predicted state is substituted in a control feedback law originally designed for the delay-free case. As a result, the stabilizing action of the delay-free control law is recovered.

Several prediction techniques are available to obtain this predicted state. The finite spectrum assignment method (Manitius and Olbrot, 1979) uses Cauchy's formula to deliver an exact prediction when all system parameters and the delay are known. This technique relies on the computation of integral terms. The introduction of dynamics in this scheme simplifies the analysis (Mondié and Michiels, 2003) and remedies robustness issues. An advantage is that no tuning is required. Another possibility is to use the so-called observer-predictors that achieve the task of observation and prediction simultaneously (Germani et al., 2002; Najafi et al., 2013). They contain a copy of the system to be controlled, thus avoiding the computation of integrals. However, they do not provide an exact prediction of the future state, hence they require a careful design of the observer-predictor gains, a task that is usually carried out via LMI techniques (Najafi et al., 2013; Lei and Khalil, 2016; Zhou et al., 2017).

Inspired by tuning formulae obtained for the controller parameters for time-delay systems or delay-based control laws (Villafuerte et al., 2013; Ramírez et al., 2016), we propose a tuning strategy based on frequency-domain techniques leading to explicit formulae that can be straightforwardly applied by the user. More precisely, we aim at tuning the predictor gains so that the prediction error, described by a delay system, achieves an exponential decay — which is limited by the delay.

Our strategy is inspired by the work of Michiels and Niculescu (2007), who show that the local optimality of the rightmost eigenvalues involves the existence of multiple roots. The recent advances in the study of *Multiplicity-Induced-Dominance* (MID) properties of quasipolynomials pave our way to carry out this task. Particularly, Balogh et al. (2022) introduce convenient formulae for the coefficients assigning multiple roots, provide useful technical properties, and prove substantial results on MID. These results follow previous contributions in this line of research, such as the study quasipolynomials of retarded type, with all coefficients free, analyzed by Mazanti et al. (2020); the case of second-order quasipolynomials addressed by Boussaada et al. (2020); as well as the partial results presented in conference papers (Boussaada et al., 2016; Boussaada and Niculescu, 2018; Boussaada et al., 2020).

In our case, the dynamics of the prediction error can be represented by a quasipolynomial where not all the coefficients are free, as in Mazanti et al. (2021). With the help of the above-mentioned MID results, we provide a new tuning method that assigns a stable dominant root to the prediction error equation.

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The contribution is organized as follows. In Section 2 we introduce the motivation for our problem and its formulation in the frequency domain. The main results of this research can be found in Section 3. In Section 4, we present a simple illustrative example. Finally, conclusions are given in Section 5.

2. PROBLEM FORMULATION

We consider control-affine systems having a constant time-delay in the input. We are particularly interested in the case in which the system is observable, uniformly with respect to the input. To simplify the exposition, we focus on single-input-single-output systems. It is well-know that, at least locally, any such system can be put in the canonical form (Gauthier and Bornard, 1980)

$$\begin{aligned} \dot{x}(t) &= Jx(t) + \varphi(x(t), u(t-h)) \\ y(x) &= Cx(t) \end{aligned} \quad , \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$, and $u(t) \in \mathbb{R}$ are the state, the output, and the input at time t, respectively. Note that the input acts on the system with a delay $h > 0$. The matrix $J \in \mathbb{R}^{n\times n}$ is a Jordan block with zero eigenvalue, $C = [1 \ 0 \ \cdots \ 0]$, and φ is triangular of the form

$$\varphi(x, u) = \begin{bmatrix} \varphi_1(x_1, u) \\ \varphi_2(x_1, x_2, u) \\ \vdots \\ \varphi_n(x_1, \dots, x_n, u) \end{bmatrix} , \tag{2}$$

nonlinear in x and affine in u.

In general, the prediction problem can be stated as follows

Problem 1. Given a system described by (1) and knowledge of $u(\tau)$ and $y(\tau)$ for $-h \le \tau \le t$, find an estimate $\hat{x}(t)$ for $x(t+h)$. By an *estimate* for $x(t+h)$, we mean a vector-valued function $\hat{x}$ such that $\lim_{t\to\infty} (\hat{x}(t) - x(t+h)) = 0$.

The problem is motivated by the following consideration. Suppose that there exists a state feedback $\kappa : \mathbb{R}^n \to \mathbb{R}$ such that the origin of

$$\dot{x}(t) = Jx(t) + \varphi(x(t), \kappa(x(t))) \tag{3}$$

is asymptotically stable. A reasonable control strategy is to enforce the law

$$u(t) = \kappa(\hat{x}(t)) \tag{4}$$

so that, if peaking phenomena do not occur in (1), the effect of the delay is compensated by the prediction and the trajectories of (1, 4) approach those of (3).

The canonical form (1) suggests the predictor

$$\dot{\hat{x}}(t) = J\hat{x}(t) + \varphi(\hat{x}(t), u(t)) + L(y(t) - C\hat{x}(t-h)) \ , \tag{5}$$

where $L = [l_1 \ l_2 \ \cdots \ l_n]^\top$ is the predictor gain, whose tuning is the main topic of this paper. The prediction error $e(t) = x(t) - \hat{x}(t-h)$ evolves according to the time-delay differential equation

$$\dot{e}(t) = Je(t) - LCe(t-h) + \Phi(x(t), \hat{x}(t-h), u(t-h)) \tag{6}$$

with $\Phi(x, \hat{x}, u) = \varphi(x, u) - \varphi(\hat{x}, u)$.

Assumption 2. The function φ in (2) is globally Lipschitz in x, uniformly in u. In other words, there exists a positive constant γ_φ such that

$$||\varphi(x, u) - \varphi(\hat{x}, u)|| \le \gamma_\varphi \, ||x - \hat{x}|| \tag{7}$$

for all $x, \hat{x} \in \mathbb{R}^n$ and all $u \in \mathbb{R}$.

The assumption is typical in the observability literature. It is stated for two reasons. On the one hand, it ensures that solutions of (1) are forward complete (no peaking phenomena arise) and, on the other hand, it opens the possibility for dominating Φ using the linear terms in (6).

Having laid down the structure (5), we can further specify our objective.

Problem 3. Choose the gain L such that the trivial solution, $e(t) \equiv 0$, is an asymptotically stable equilibrium of (6).

The solution to Problem 3 is simple when $h = 0$. Indeed, consider the operator

$$\mathcal{L} : e \mapsto (J - LC)e \ .$$

Note that the pair (J, C) is observable so, as long as symmetry with respect to complex conjugation is respected, the spectrum can be arbitrarily assigned by a proper choice of L. Because of Assumption 2, the nonlinear map Φ can be dominated by $\mathcal{L}$ and the trivial solution of (6) can be made asymptotically stable by placing a strictly negative spectral abscissa with a sufficiently large absolute value.

Problem 3 becomes more challenging when $h > 0$. Firstly, the state of (6) is no longer an n-dimensional vector e, but a function $e_t : \theta \mapsto e(t + \theta)$, $\theta \in [-h, 0]$ (Kharitonov, 2013). The spectrum of the operator

$$\mathcal{D} : e_t \mapsto Je(t) - LCe(t-h) \tag{8}$$

is now infinite and of course cannot be arbitrarily assigned using the n-dimensional gain L. However, we will show below that there are enough degrees of freedom for placing the spectrum of $\mathcal{D}$ inside the left half-plane. Unfortunately, the delay imposes a limit on how far to the left can the spectrum be pushed, and this in turn establishes a limit on how large the Lipschitz constant γ_φ is allowed to be.

The characteristic function of (8) is

$$D_L(s) = P(s) + Q_L(s)e^{-sh} \ , \tag{9}$$

where

$$P(s) = s^n \quad \text{and} \quad Q_L(s) = \begin{bmatrix} s^{n-1} & s^{n-2} & \cdots & s & 1 \end{bmatrix} \cdot L \ .$$

We say that a root s^* of $D_L(s)$ is *dominant* if

$$\Re(s^*) = \max \{\Re(s) \mid D_L(s) = 0\} \ . \tag{10}$$

Thus, solving Problem 3 requires us to find L such that the dominant roots of $D_L(s)$ lie inside the left half-plane.

3. MAIN RESULT

In this section, we solve Problem 3 for sufficiently small γ_φ [1]. Motivated by Theorem 7.6 in (Michiels and Niculescu, 2007) we search for gains L such that $D_L(s)$ has roots with multiplicity equal to $n+1$. In contrast with the case in studied by Mazanti et al. (2021), the roots of maximal multiplicity are not unique and not necessarily dominant. However, we will show with the help of the results by Balogh et al. (2022) that there is one specific L^* for which the root with multiplicity $n + 1$ is indeed dominant and that, fortunately, it lies on the left half-plane.

[1] The solution for arbitrary γ_φ will be presented elsewhere.

3.1 Multiple-root assignment

We begin by characterizing all the roots of $D_L(s)$ that can be multiply assigned by L. Finding a root with multiplicity $n+1$ is equivalent to finding a solution σ for

$$D_L(s)\Big|_{s=\sigma} = 0\,, \quad \frac{\mathrm{d}D_L(s)}{\mathrm{d}s}\Big|_{s=\sigma} = 0\,, \quad \cdots\,, \quad \frac{\mathrm{d}^n D_L(s)}{\mathrm{d}s^n}\Big|_{s=\sigma} = 0\,. \tag{11}$$

In our special case, the notation introduced by Balogh et al. (2022) allows an elegant rewriting of conditions (11) in matrix form. For each $k = 0, 1, \ldots, n$, we implicitly define $R_k(s,h)$, a polynomial in s and h given by

$$R_k(s,h)e^{sh} = \frac{\mathrm{d}^k P(s)e^{sh}}{\mathrm{d}s^k}\,. \tag{12}$$

Multiplying (11) by e^{sh} and using the implicit definition (12), we can express (11) as

$$M(\sigma)\hat{I}L + R(\sigma,h)e^{\sigma h} = 0 \tag{13a}$$

$$R_n(\sigma,h)e^{\sigma h} = 0 \tag{13b}$$

with

$$R(s,h) = \begin{bmatrix} R_0(s,h) \\ R_1(s,h) \\ \vdots \\ R_{n-1}(s,h) \end{bmatrix}, \quad \hat{I} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix},$$

and $M(s) = [m_{ij}(s)]$ some $n \times n$ matrix whose entries are real polynomials in s. For $P(s) = s^n$, $R_k(s,h)$ and $M(s)$ take the explicit forms

$$R_k(s,h) := \sum_{i=1}^{k+1} \binom{n}{k-i+1} \frac{k!}{(i-1)!} s^{n-k+i-1} h^{i-1} \tag{14}$$

and

$$m_{ij}(s) = \begin{cases} \dfrac{(j-1)!}{(j-i)!} s^{j-i} & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases}, \quad i,j = 1, \ldots, n\,.$$

Notice that (13b) is independent of L, which allow us to proceed in two steps: First solve (13b) for σ and then solve (13a) for L. We now focus on (13b).

Lemma 4. The two-variable polynomial $R_n(s,h)$, defined in (13b), can be written as a single-variable polynomial: $R_n(s,h) = q(sh)$ with

$$q(r) := \sum_{j=0}^{n} \binom{n}{j} \frac{n!}{j!} r^j\,. \tag{15}$$

Proof. It follows from (12) that

$$R_n(s,h) = \sum_{i=1}^{n+1} \binom{n}{n-i+1} \frac{n!}{(i-1)!} (sh)^{i-1}\,.$$

Relabeling the summation index as $j = i-1$ we see that

$$R_n(s,h) = \sum_{j=0}^{n} \binom{n}{n-j} \frac{n!}{j!} (sh)^j\,.$$

The symmetry $\binom{n}{n-j} = \binom{n}{j}$ finally gives $R_n(s,h) = q(sh)$. □

The following theorem characterizes the roots that can be multiply assigned to $D_L(s)$. The proof is constructive, giving explicit formulae for the gains that assign such roots.

Theorem 5. Consider the polynomial $q(r)$ defined in (15) and let

$$\mathcal{S} = \{s \in \mathbb{R} \mid q(sh) = 0\}\,. \tag{16}$$

If σ is a root with multiplicity $n+1$ of the quasipolynomial $D_L(s)$ defined in (9), then $\sigma \in \mathcal{S}$. Conversely, for any $\sigma \in \mathcal{S}$, we can find a gain $L = L(\sigma) \in \mathbb{R}^n$ such that σ is a root with multiplicity $n+1$ of $D_L(s)$. Moreover, such gain is given by

$$L(\sigma) = \operatorname{diag}\left(\tfrac{1}{h}, \tfrac{1}{h^2}, \ldots, \tfrac{1}{h^n}\right) \bar{L}(\sigma h)\,, \tag{17}$$

where $\bar{L}(r) = \begin{bmatrix} \bar{l}_1(r) & \bar{l}_2(r) & \cdots & \bar{l}_n(r) \end{bmatrix}^\top$ is defined by

$$\bar{l}_k(r) = \sum_{j=n-k+1}^{n} \sum_{i=1}^{j} \left[(-1)^{n+j+k} \binom{n}{i-j} \cdot \binom{j-1}{n-k} \frac{1}{(i-1)!} r^{i-1} \right] r^k e^r \tag{18}$$

for $k = 1, 2, \ldots, n$.

Proof. If σ is a characteristic root with multiplicity $n+1$, then σ necessarily satisfies (13b) so, by Lemma 4, $r = \sigma h$ is a root of $q(r)$ and $\sigma \in \mathcal{S}$.

Now, suppose that $\sigma \in \mathcal{S}$. Again by Lemma 4, we know that σ satisfies (13b). It only remains to show that we can find L that satisfies (13a). We have

$$M(\sigma)\hat{I}L + R(\sigma,h)e^{\sigma h} = 0\,. \tag{19}$$

Note that $\hat{I}^{-1} = \hat{I}$ and the determinant of the triangular matrix $M(s)$ is a nonzero constant (a unit in the ring $\mathbb{R}[s]$). Thus, the inverse matrix $M^{-1}(s)$ exists; it is actually given by

$$m_{i,j}^{-1}(s) = \begin{cases} (-1)^{j+i} \dfrac{1}{(i-1)!(j-i)!} s^{j-i} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases},$$

$i,j = 1, 2, \ldots, n$. We can solve the equation (19) for L as

$$L(\sigma) = -\hat{I}M^{-1}(\sigma)R(\sigma,h)e^{\sigma h}\,. \tag{20}$$

Finally, we can verify that (17, 18) is an explicit rewriting of (20). □

3.2 Stability of the multiple roots

The multiply assignable roots in $\mathcal{S}$ are useful only if they indeed turn out to be dominant and stable. We now address the latter condition.

Lemma 6. The polynomial $q(r)$, defined in (15), has n negative and distinct real roots.

Proof. The proof relies on Sturm's Theorem (Meserve, 1982, pp. 160-164) which allows to determine the number of distinct real roots of any given real polynomial on any interval of the real axis. To apply the Sturm Theorem, it is necessary to compute the Euclidean algorithm for $q(r)$ and $g_1(r)$, where $g_1(r)$ is defined as

$$\frac{\mathrm{d}q(r)}{\mathrm{d}r} = c_1 g_1(r)$$

where c_1 is a constant and $g_1(r)$ is a monic polynomial. The division sequence between $q(r)$ and $g_1(r)$ is called the Sturm sequence and is made of all dividends in the Euclidean algorithm:

$$\{q(r),\, g_1(r), \ldots, g_n(r)\}\,.$$

The Sturm sequence is not straightforward to obtain, however, computing the dividends we see that

$$g_i(r) = \sum_{j=0}^{n-i} \binom{n-i}{j} \frac{(n-i+1)!}{(j+1)!} r^j, \quad i = 1, 2, \ldots, n. \quad (21)$$

The Sturm Theorem states that the number of roots on a given interval is equal to the difference of sign changes that occur in Sturm's sequence when evaluated at the interval endpoints.

We apply the Sturm's Theorem to the interval $(-\infty, 0]$. Let v_0 be the number of sign variations of the Sturm sequence at $r = 0$. We have $q(0) = n!$ and

$$g_i(0) = (n-i+1)! \ , \quad i = 1, \ldots, n \ .$$

Clearly, there are no sign variations at $r = 0$. Thus, $v_0 = 0$.

The number of sign variations at $-\infty$ is denoted by $v_{-\infty}$. We have

$$\begin{aligned}\lim_{r\to-\infty} q(r) &= \lim_{r\to-\infty} \sum_{j=0}^{n} \binom{n}{j} \frac{n!}{j!} r^j \\ &= \lim_{r\to-\infty} r^n \left[1 + \sum_{j=0}^{n-1} \binom{n}{j} \frac{n!}{j!} \frac{r^j}{r^n} \right] .\end{aligned}$$

Since the second term in brackets tends to zero as $r \to -\infty$, it follows that

$$\lim_{r\to-\infty} q(r) = (-1)^n \infty \ .$$

Hence, when $r \to -\infty$, the sign of $q(r)$ is positive if n is even and negative if n is odd. Similarly,

$$\begin{aligned}\lim_{r\to-\infty} g_i(r) &= \lim_{r\to-\infty} \sum_{j=0}^{n-i} \binom{n-i}{j} \frac{(n-i+1)!}{(j+1)!} r^j \\ &= \lim_{r\to-\infty} r^{n-i} \left[1 + \sum_{j=0}^{n-i-1} \binom{n-i}{j} \cdot \right. \\ &\qquad\qquad \left. \frac{(n-i+1)!}{(j+1)!} \frac{r^j}{r^{n-i}} \right] .\end{aligned}$$

Hence,

$$\lim_{r\to-\infty} g_i(r) = (-1)^{n-i} \infty, \quad i = 1, 2, \ldots, n \ .$$

Thus, as $r \to -\infty$, the Sturm sequence has n sign variations.

By Sturm's Theorem, the polynomial $q(r)$ has $v_{-\infty} - v_0 = n$ roots on the interval $(-\infty, 0]$. □

The following theorem is a direct consequence of Lemma 6 and the definition of $\mathcal{S}$ in (16).

Theorem 7. Consider the set of multiply assignable roots $\mathcal{S}$ given in (16). All such roots are real and stable, that is,

$$\sigma < 0 \ , \quad \sigma \in \mathcal{S} \ .$$

3.3 Dominance of the multiple roots

Given that we now know that all the roots that can be multiply assigned are stable, a natural question is: Which of these roots — if any — will turn out to be dominant? Before giving a partial answer to this question, allow us to present a couple of technical lemmas.

Lemma 8. Let the set $\mathcal{S}$ be given by (16) and define

$$\sigma^* = \max \mathcal{S} \ . \quad (22)$$

Then, $R_n(\sigma^*, \theta)$ is a strictly decreasing function of θ on the interval $[0, h)$. Moreover,

$$R_n(\sigma^*, \theta) > 0 \ ,$$

also on the interval $[0, h)$.

Proof. From $q(r)$ defined in (15), note that $q(\sigma^* h) = 0$ and $q(0) = n!$. Since there are no roots of $q(r)$ on the interval

$$I = (\sigma^* h, 0] \ ,$$

$q(r)$ remains positive on such interval.

Recall that, the derivative of $q(r)$ with respect to r is $c_1 g_1(r)$ with $g_1(r)$ defined in (21) and $c_1 = n$. By Rolle's Theorem (Apostol, 1974), each of the $n-1$ roots of $g_1(r)$ are located between two roots of $q(r)$; hence, there are no roots of $g_1(r)$ on I and the sign of $g_1(r)$ remains constant on I.

Since $g_1(0) = n! > 0$, we conclude that $q(r)$ is an increasing function of r on I. Since σ^* is negative and $R_n(\sigma^*, \theta) = q(\sigma^* \theta)$ (Lemma 4), we conclude that $R_n(\sigma^*, \theta)$ is a decreasing function of θ on the interval $[0, h)$. Finally, since $R_n(\sigma^*, 0) = n!$ and $R_n(\sigma^*, h) = 0$, we have that $R_n(\sigma^*, \theta) > 0$ for all $\theta \in [0, h)$. □

Lemma 9. Let σ belong to the set $\mathcal{S}$ given by (16) and let L be given by (17). Then, $D_{L(\sigma)}(s)$ admits the factorization

$$D_{L(\sigma)}(s) = \frac{1}{n!} (s - \sigma)^{n+1} \int_0^h R_n(\sigma, \theta) e^{-(s-\sigma)\theta} \mathrm{d}\theta \ . \quad (23)$$

Proof. Following the factorization shown in (Mazanti et al., 2021) but applied to $D_{L(\sigma)}(s)$ to isolate n roots, and then integrating by parts to isolate one more, we can see that (23) is obtained. □

We are now ready to present our main result, which asserts that it is possible to assign σ^* as a dominant root of the characteristic quasipolynomial.

Theorem 10. (Main result). Let σ^* be given as in (22) and define

$$L^* = L(\sigma^*)$$

with $L(\sigma)$ given by (17). Then, σ^* is a dominant negative root with multiplicity $n+1$ of $D_{L^*}(s)$.

Proof. We have already established multiplicity (Thm. 5) and stability (Thm. 7). To prove the dominancy of σ^*, we evaluate the factorization (23),

$$D_{L^*}(s) = \frac{1}{n!} (s - \sigma^*)^{n+1} \int_0^h R_n(\sigma^*, \theta) e^{-(s-\sigma^*)\theta} \mathrm{d}\theta \ .$$

The roots of the polynomial $(s - \sigma^*)^{n+1}$ are indeed $n+1$ roots located at σ^*. It remains to show that the multiplicand

$$f(s) = \int_0^h e^{-(s-\sigma^*)\theta} R_n(\sigma^*, \theta) \mathrm{d}\theta$$

cannot have roots with real part greater than σ^*.

First, we prove that $f(s)$ has no real roots. For the sake of contradiction, assume that $s = a$, $a \in \mathbb{R}$. By Lemma 8, we have that

$$R_n(\sigma^*, \theta) > 0 \ , \quad \theta \in [0, h).$$

As $e^{-(a-\sigma^*)\theta} > 0$, the integrand is strictly positive in $[0, h]$, hence $f(a)$ cannot vanish, regardless of the value of a.

Next, we turn our attention to the case of complex conjugate roots. Assume that $a \pm ib$, $a > \sigma^*$, is a pair of complex conjugate roots of $f(s)$. Without loss of generality, suppose that $b > 0$. We have

$$f(a+ib) = \int_0^h e^{-(a-\sigma^*)\theta} R_n(\sigma^*, \theta) e^{ib\theta} \mathrm{d}\theta \,.$$

By Euler's identity, it follows that

$$\begin{aligned} \Re\left(f(a+ib)\right) &= \int_0^h e^{-(a-\sigma^*)\theta} R_n(\sigma^*, \theta) \cos(b\theta) \mathrm{d}\theta = 0 \,, \\ \Im\left(f(a+ib)\right) &= \int_0^h e^{-(a-\sigma^*)\theta} R_n(\sigma^*, \theta) \sin(b\theta) \mathrm{d}\theta = 0 \,. \end{aligned} \tag{24}$$

Applying the change of variable $\theta = \zeta/b$ to (24) gives

$$\Im\left(f(a+ib)\right) = \frac{1}{b} \int_0^{bh} e^{-(a-\sigma^*)\frac{\zeta}{b}} R_n\left(\sigma^*, \tfrac{\zeta}{b}\right) \sin(\zeta) \mathrm{d}\zeta \,.$$

Let $\alpha \in \mathbb{Z}$ be the quotient of bh over 2π, and let $0 \le \beta < 2\pi$ be the remainder, i.e., $bh = 2\pi\alpha + \beta$. The integral can be separated as

$$\Im\left(f(a+ib)\right) = \frac{1}{b}(g_\alpha + g_\beta) \,,$$

where

$$g_\alpha = \int_0^{2\pi\alpha} e^{-(a-\sigma^*)\frac{\zeta}{b}} R_n\left(\sigma^*, \tfrac{\zeta}{b}\right) \sin(\zeta) \mathrm{d}\zeta$$

and

$$g_\beta = \int_{2\pi\alpha}^{2\pi\alpha+\beta} e^{-(a-\sigma^*)\frac{\zeta}{b}} R_n\left(\sigma^*, \tfrac{\zeta}{b}\right) \sin(\zeta) \mathrm{d}\zeta \,.$$

Recall that, given any $k \in \mathbb{Z}$, $\sin(\zeta)$ is positive for $\zeta \in (2\pi k, 2k\pi + \pi)$ and negative for $\zeta \in (2k\pi + \pi, 2\pi(k+1))$. We divide the integral g_α according to the positive and negative values of the sine function,

$$\begin{aligned} g_\alpha = \sum_{k=0}^{\alpha-1} \Bigg\{ &\int_{2k\pi}^{2k\pi+\pi} e^{-(a-\sigma^*)\frac{\zeta}{b}} R_n(\sigma^*, \tfrac{\zeta}{b}) \sin(\zeta) \mathrm{d}\zeta \\ &+ \int_{2k\pi+\pi}^{2(k+1)\pi} e^{-(a-\sigma^*)\frac{\zeta}{b}} R_n(\sigma^*, \tfrac{\zeta}{b}) \sin(\zeta) \mathrm{d}\zeta \Bigg\} \,. \end{aligned}$$

Lemma 8 states that $R_n(\sigma^*, \theta) > 0$ is strictly decreasing for $\theta \in [0, h)$. Since $a > \sigma^*$, the exponential term $e^{-(a-\sigma^*)\theta}$ is also strictly decreasing. Thus, the first (positive) integrand inside the braces dominates the second (negative) one. Hence, the term into braces is strictly positive $\forall k$.

Let us now consider the integral g_β. If $\beta \in (0, \pi)$, the sine is positive and so is g_β. If $\beta \in (\pi, 2\pi)$, we further divide g_β as

$$\begin{aligned} g_\beta = &\int_{2\pi\alpha}^{2\pi\alpha+\pi} e^{-(a-\sigma^*)\frac{\zeta}{b}} R_n(\sigma^*, \tfrac{\zeta}{b}) \sin(\zeta) \mathrm{d}\zeta \\ &+ \int_{2\pi\alpha+\pi}^{2\pi\alpha+\beta} e^{-(a-\sigma^*)\frac{\zeta}{b}} R_n(\sigma^*, \tfrac{\zeta}{b}) \sin(\zeta) \mathrm{d}\zeta \,. \end{aligned}$$

Arguing as before, we see that g_β is non-negative. As a consequence, $\Im(f(a \pm ib))$ is strictly positive and $f(a \pm ib)$ cannot vanish when $a > \sigma^*$. We conclude that neither real nor complex roots with real part greater than σ^* exist; thus, σ^* is a dominant root. □

A direct implication of Theorem 10 is the following.

Corollary 11. The characteristic quasipolynomial $D_{L^*}(s)$ is stable.

We have now completed our task. Theorem 10 provides the parameter values of the gain L^* that assigns the root of maximal multiplicity located in the open left half-plane.

Remark 12. It is worthy of mention that we can interpret the above result in terms of (Balogh et al., 2022, Proposition 2). We have established that $R_n(s_n, \theta)$ is decreasing on the interval $[0, h]$. Thus in view of the property

$$\frac{\partial R_n(s_n, \theta)}{\partial \theta} = n R_{n-1}(s_n, \theta)$$

proved in (Balogh et al., 2022), it follows that the derivative of $R_n(s_n, \theta)$ is negative for $\theta \in [0, h)$, i.e., $R_{n-1}(s_n, \theta)$ is negative over $[0, h)$. Then it follows from (Balogh et al., 2022, Proposition 2) that the root is dominant.

A significant advantage of this tuning method is that the predictor gain is obtained via the determination of the roots of a polynomial of degree n followed by the use of explicit formulae. It is summarized as follows:

- Compute the n roots of $q(r)$ defined in (15).
- Select the the rightmost root r^* and set
$$\sigma^* = \frac{r^*}{h} \,.$$
- Compute the predictor gain $L^* = L(\sigma^*)$ according to formulae (17, 18).

4. ILLUSTRATIVE EXAMPLE

We illustrate our observer-predictor design for the control of the simple pendulum equation (Khalil, 2002, pp. 5-6) with an input delay described by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{g}{l} \sin(x_1(t)) + \frac{1}{ml^2} u(t-h) \\ y(t) &= x_1(t), \end{aligned} \tag{25}$$

where $x_1(t)$ and $x_2(t)$ are the position and velocity of the pendulum, respectively, $u(t)$ is the input signal, $y(t)$ is the output signal and h is the input delay; $l = 1$ m is the length of the pendulum, $m = 1$ kg is its mass and $g = 9.81$ m/s^2. System (25) is of the form (1) with

$$\varphi(x(t), u(t-h)) = \left[0,\ \frac{g}{l} \sin(x_1(t)) + \frac{1}{ml^2} u(t-h)\right]^\top .$$

As sketched in Fig. 1, the control law $u(t) = \kappa(x(t))$, where

$$\kappa(x(t)) := ml^2 \left(-\frac{g}{l} \sin x_1(t) - k_1 x_1(t) - k_2 x_2(t)\right), \tag{26}$$

with $k_1 = 20$ and $k_2 = 100$, stabilizes the delay-free system (25), but the closed-loop (25, 26) becomes unstable for a delay $h = 0.065$. Now, consider introducing the following predictor in the closed-loop,

$$\begin{aligned} \dot{\hat{x}}_1(t) &= \hat{x}_2(t) + l_1 \left(y(t) - \hat{x}_1(t-h)\right) \\ \dot{\hat{x}}_2(t) &= \frac{g}{l} \sin(\hat{x}_1(t)) + \frac{1}{ml^2} \kappa(\hat{x}(t)) + l_2 \left(y(t) - \hat{x}_1(t-h)\right) \end{aligned} \tag{27}$$

To tune the prediction gain, we first compute the n roots of $q(r) = r^2 + 4r + 2$. The roots are $r_1 = -2 - \sqrt{2}$ and $r_2 = -2 + \sqrt{2}$. We chose the rightmost root r_2 and compute

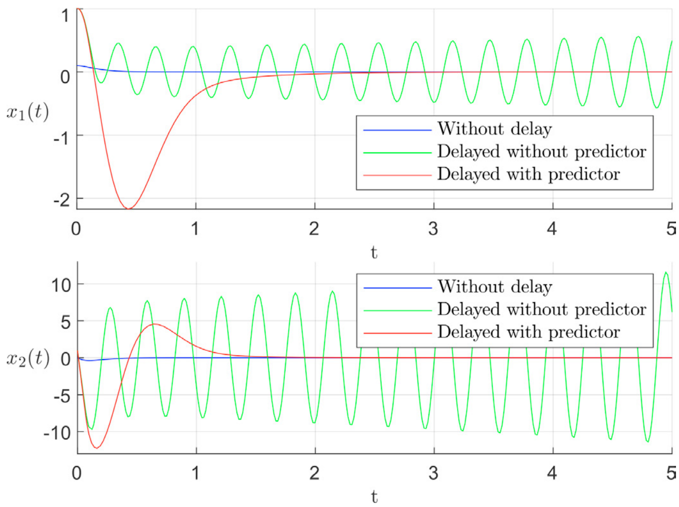

Fig. 1. The closed-loop (25, 26) is stable without delay, but losses stability with $h = 0.065$. Nevertheless, the closed-loop recovers the stability with the predictor (27) and the control $u(t) = \kappa(\hat{x}(t))$.

$$\sigma^* = \frac{-2 + \sqrt{2}}{h} = -9.0121.$$

Finally, we substitute σ^* into the explicit formula (17) to compute the predictor gains

$$l_1 = 7.0948 \ , \quad l_2 = 18.7272 \ .$$

We observe in Fig. 1 that the closed-loop recovers stability in presence of the delay.

5. CONCLUDING REMARKS

A tuning methodology for the prediction error equation of systems with input delay achieving exponential stability is presented. It amounts to finding the roots of a polynomial of the system's degree and substituting the rightmost one into simple algebraic formulae. We introduce several novel techniques for proving the root dominancy and stability, which we believe are of interest in their own right. A pending query is whether or not the parameter's choice achieves the maximal exponential decay of the prediction error.

The Lyapunov stability analysis of the closed-loop nonlinear system (1,4,5) and the determination of the maximum admissible Lipchitz constant γ_φ will be presented in the near future. A promising direction of research is the use of sub-predictors allowing a larger exponential decay of the prediction error and, consequently, a larger Lipchitz constant for the nonlinear term.

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