Implicit and explicit representations of continuous-time port-Hamiltonian systems

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Abstract

Implicit and explicit representations of smooth, finite-dimensional port-Hamiltonian systems are studied from the perspective of their use in numerical simulation and control design. Implicit representations arise when a system is modeled in Cartesian coordinates and when the system constraints are applied in the form of additional algebraic equations. Explicit representations are derived when generalized coordinates are used. A relationship between the phase spaces for both system representations is derived in this article, justifying the equivalence of the representations in the sense of preserving their Hamiltonian functions as well as their Hamiltonian symplectic forms, ultimately resulting in the same Hamiltonian flow.

Keywords: Port-Hamiltonian systems, nonlinear implicit systems, modeling of physical systems.

1. Introduction

Hamiltonian systems form an important class of conservative systems. They appear in many disciplines, including mechanical engineering, circuit theory and astronomy. The Hamiltonian point of view is worth particular attention because it allows to solve a great many of mechanical problems which do not easily lend themselves to solutions. An example is the problem of attraction by two stationary centers [1]. Hamiltonian methods are also invaluable in derivation of approximate methods in perturbation theory and for clarifying the character of motion of systems in celestial and statistical mechanics.

The class of Hamiltonian systems was extended in [2, 3] to include open systems that can interact with the environment via a set of inputs and outputs, termed *ports*, giving rise to port-Hamiltonian (PH) systems. These extended models reveal the passivity properties of the corresponding system, making them particularly well suited for the design of passivity-based control (PBC) feedback laws.

Complex dynamic systems are preferably modeled in terms of simpler interconnected subsystems, typically represented by systems of ordinary differential equations (ODEs) expressed in Cartesian space coordinates. Assembling the subsystems amounts to imposing a set of algebraic constraints, such that the resulting implicit model becomes a system of differential-algebraic equations (DAEs).

Under reasonable assumptions, a DAE can be reformulated as an equivalent ODE on a manifold [4, 5]. In the language of analytical mechanics, the local coordinates on the manifold in the explicit system representation are called *generalized coordinates*. These coordinates automatically satisfy the system constraints and the number of degrees of freedom of the system is equal to the dimension of the submanifold on which the system evolves. Since in explicit ODE system representations the constraints and associated constraint forces no longer appear, these representations are preferred in the analysis of general mechanical systems [6, 7] and in many problems of circuit theory see [8, 9]. The main drawback of this framework is that the resulting Hamiltonian (energy) function complicates substantially.

Implicit DAE system representations [10, 11] are of higher dimension and, depending on the particular application, may require to solve for variables defined implicitly. On the other hand, the corresponding Hamiltonian functions have simpler expressions. More precisely, it is possible to split the Hamiltonian functions in two terms: one depending on the velocities only (the kinetic energy) and one depending on the positions only (the potential energy), i.e., the Hamiltonians are *separable*. This property has been largely exploited in the context of numerical simulations, resulting in self-correcting numerical simulation algorithms [12] and discrete-time sampled-data PH models. When applying the interconnection and damping assignment PBC methodology [13] to a PH system,

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the resulting partial differential equations are simplified considerably if the kinetic energy does not depend on the positions, so there are possible advantages in using implicit representations in a control context as well (other advantages are suggested in [11]).

1.1. Motivation

For Hamiltonian systems, both DAE and ODE models have been extensively studied [6, 10, 12], but little has been said about the correspondence between the associated phase spaces and system flows. It has been briefly suggested in [7, 14] that, formally, the manifold on which the ODE is defined can be regarded as an embedded submanifold of the original Euclidean configuration space, but a detailed analysis about the mapping between phase spaces has not been carried out. A rigorous discussion of these relationships is presented here.

Some properties of PH systems that are important from the control perspective (such as passivity) have been thoroughly addressed within the ODE framework only [13]. Passivity has been discussed within the DAE framework from an algebraic perspective, using the notion of Dirac structure [3], but a geometric treatment is still needed.

1.2. Contributions

We provide a map $(\imath, \jmath^*)$ from the phase space of the ODE system model to the phase space of the DAE system model. The map is shown to commute with the Legendre transform (Propositions 6 and 7), justifying the use of DAE and ODE models on an equal footing. For systems with inputs, we show how the control vector fields are mapped from one representation to the other. This is summarized on Table 3.1.

Proposition 14 provides simple conditions for assessing the passivity of PH systems defined implicitly. Proposition 15 shows how the symplectic form changes in the presence of external inputs (this computation is new, both in the explicit and implicit frameworks).

1.3. Paper structure

The purpose of Section 2 is to establish a concrete relation between phase spaces. Section 2.1 presents conditions for the configuration space to be embeddable in a higher-dimensional ambient space. Sections 2.2 and 2.3 use this embedding to relate the tangent and cotangent bundles of the configuration space to their corresponding subbundles of the ambient space. Sections 2.4 and 2.5 recall the Legendre transform and Hamilton equations. Energy conservation and symplecticity are discussed in Section 2.6.

In Section 3 we consider systems with inputs and discuss their effect on the properties of energy conservation and symplecticity. Special emphasis is given to implicit representations.

Conclusions and future work are given in Section 4.

2. Hamiltonian Systems as ODEs and DAEs

2.1. System Configuration Spaces

The class of Hamiltonian systems considered here is restricted to mechanical finite-dimensional systems with holonomic constraints that evolve in continuous time. Systems of this type typically consist of M rigid bodies held together as one structure by the action of constraint forces. The position of each of the rigid bodies can be unambiguously described in terms of the position of its center of mass and its orientation in space The *configuration space* of a spatial system as a whole can be thought to be a subset of a Cartesian space of dimension $n = 6M$. In the plane, $n = 3M$.

It is assumed for simplicity that the holonomic system constraints are expressed by smooth (C^∞) *independent* functions $g : \mathbb{R}^n \to \mathbb{R}^k$, with $k \leq n$, that do not depend on time. The configuration space of the constrained system as a whole in the implicit DAE form is then the level set $\mathbb{G} := g^{-1}(0) = \{r \in \mathbb{R}^n \mid g(r) = 0\}$, where $g^{-1}(0)$ is the inverse image of $0 \in \mathbb{R}^k$. Functional independence of the constraints is expressed in terms of the *rank condition.*

Assumption 1. *The rank of g is equal to k at all points of the set $g^{-1}(0)$.*

Under this assumption, the value $0 \in \mathbb{R}^k$ is called a *regular value* of g. The level set $g^{-1}(0)$ is a *regular level set* of g and g is said to be a *defining map* for $\mathbb{G}$ [15, p. 183–184]. It follows that $\mathbb{G}$ is a closed embedded submanifold of $\mathbb{R}^n$ [15, Cor. 8.10, p. 182]. The Constant-Rank Level Set Theorem [15, Th. 8.8, p. 182] further specifies the dimension of $\mathbb{G}$ to be $o = n - k$. This motivates the following definition.

Definition 2. *The* number of degrees of freedom *of a system with $\mathbb{G}$ as its configuration space is defined as the dimension of $\mathbb{G}$. The coordinates delivered by the charts on $\mathbb{G}$ are called* generalized coordinates *for the system.*

Since $\mathbb{G}$ is an embedded submanifold of $\mathbb{R}^n$, there exists an inclusion map $\imath : \mathbb{G} \hookrightarrow \mathbb{R}^n$ (embedding) with $\operatorname{rank}(\imath) = \dim \mathbb{G}$ and which is a homeomorphism onto $\imath(\mathbb{G}) \subset \mathbb{R}^n$ in the subspace topology on $\mathbb{R}^n$. Such an inclusion map $\imath(a) = a$ (one-to-one), sends each element a of $\mathbb{G}$ to a treated as an element of $\mathbb{R}^n$ and has different expressions in different coordinates on $\mathbb{G}$ and on $\mathbb{R}^n$. In practical situations, the geometric properties of the system under study often suggest a selection of coordinates on which the inclusion map $\imath : \mathbb{R}^o \hookrightarrow \mathbb{R}^n$ satisfies the conditions for a topological embedding and such that $g \circ \imath \equiv 0$.

Henceforward, the Cartesian coordinates on $\mathbb{R}^n$ will be denoted by $r = (r^1, \ldots, r^n)$ and the generalized coordinates of $\mathbb{G}$ will be denoted by $q = (q^1, \ldots, q^o)$. It follows from the foregoing discussion that, in generalized coordinates, $r = \imath(q)$. It is natural to ask how dynamical system models in the alternative coordinates $(r^1, \ldots, r^n)$ and $(q^1, \ldots, q^o)$ can be constructed and in which sense are these system models equivalent. The domains in which to express the Lagrangian and Hamiltonian functions are, respectively, the tangent and the cotangent bundles of the configuration space. The relations of these functions in the two representations have to be addressed first.

2.2. The Tangent Bundles and the Lagrangians

Let $T_q\mathbb{G}$ denote the tangent space to the manifold $\mathbb{G}$ at a point $q \in \mathbb{G}$ and let $T\mathbb{G} = \coprod_{q\in\mathbb{G}} T_q\mathbb{G}$ be the tangent bundle of $\mathbb{G}$. The flow of a dynamical system evolving on the manifold $T\mathbb{G}$ is then determined by application of Hamilton's Principle, which states that the action associated with a system trajectory over any interval of time $[t_1, t_2]$ has a stationary value that is expressed by

$$\hat{L} : T\mathbb{G} \to \mathbb{R} \;, \quad \delta \int_{t_1}^{t_2} \hat{L}(q, \dot{q})\mathrm{d}t = 0 \;,$$

where δ denotes the variation of the action functional over the class of smooth trajectories $q(t)$ with fixed endpoints. The Lagrangian $\hat{L}$ represents the difference between the kinetic and potential energies of the system, i.e. $\hat{L} = \hat{K} - \hat{V}$, expressed in the generalized coordinates $(q, \dot{q})$ on $T\mathbb{G}$ induced by the coordinates on $\mathbb{G}$. The tangent bundle $T\mathbb{G}$ is assumed to be equipped with a Riemannian metric, that is, a symmetric, bilinear, positive-definite form $\langle\langle\cdot,\cdot\rangle\rangle$ on $T\mathbb{G}$, so that $\hat{K} = \frac{1}{2}\langle\langle\dot{q}, \dot{q}\rangle\rangle$. The potential energy is a function $\hat{V} : \mathbb{G} \to \mathbb{R}$.

Now, let $T_r\mathbb{R}^n$ denote the tangent space to $\mathbb{R}^n$ at r and let $T\mathbb{R}^n$ be the tangent bundle of $\mathbb{R}^n$. The constrained Hamilton's Principle [16, p. 103], as stated in the Cartesian coordinates, calls for the stationarity of the constrained Lagrangian

$$L : T\mathbb{R}^n \to \mathbb{R} \;, \quad \delta \int_{t_1}^{t_2} \left(L(r, \dot{r}) + \sum_{i=1}^{k} \lambda_i g^i(r) \right) \mathrm{d}t = 0 \;,$$

where the variations are also taken over the class of paths $r(t)$ with fixed endpoints and where L satisfies $\hat{L} = L \circ \varphi$ with $\varphi := (\imath, \imath_*)$ a bundle map from $T\mathbb{G}$ to $T\mathbb{R}^n$. The necessary conditions for stationarity take the usual form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{r}} L(r, \dot{r}) - \frac{\partial}{\partial r} L(r, \dot{r}) + \sum_{i=1}^{k} \lambda_i \frac{\partial}{\partial r} g^i(r) = 0 \;.$$

The flow of the system, being constrained, does not evolve on the whole $T\mathbb{R}^n$, it rather evolves on a subbundle (a collection of linear subspaces of $T_{\imath(q)}\mathbb{R}^n$). We now provide a way to characterize such subspaces. Intuitively, these should be identifiable with $T_q\mathbb{G}$. More precisely, let $\imath_* : T_q\mathbb{G} \to T_{\imath(q)}\mathbb{R}^n$ be the push-forward by $\imath$. The desired subspaces can be simply written as $\imath_*(T_q\mathbb{G})$, but we wish to characterize them without recourse to an explicit representation for $\mathbb{G}$. This can be done in terms of derivations as follows. The image $\tilde{X} = \imath_*(X) \in T_{\imath(q)}\mathbb{R}^n$ of a vector $X \in T_q\mathbb{G}$ operates as $\tilde{X}h = (\imath_* X)h = X(h \circ \imath) = X(h_{|\mathbb{G}})$ on smooth functions h on $\mathbb{R}^n$. Here, $h_{|\mathbb{G}}$ is the restriction of h to $\mathbb{G}$. Note that the image of $T_q\mathbb{G}$ is equal to

$$\imath_*(T_q\mathbb{G}) = \{X \in T_{\imath(q)}\mathbb{R}^n \,|\, Xh = 0 \,, h \in C^\infty \,,\; h_{|\mathbb{G}} \equiv 0\} \;,$$

see [15, Prop. 8.5, p. 178]. Since the map $g = (g^1, \ldots, g^k)$ is smooth and vanishes on $\mathbb{G}$, the desired subspace is

$$\imath_*(T_q\mathbb{G}) = \{X \in T_{\imath(q)}\mathbb{R}^n \,|\, Xg^i = 0 \text{ for } i = 1, \ldots, k\} \quad (1)$$

(note that the right-hand side only depends on the defining map).

The following assumption about the convexity of the Lagrangian function is standard because it guarantees adequate regularity (of class at least $\mathcal{C}^2$) of the extremals of the associated action functional; again see [16].

Assumption 3. *The Hessian matrix*

$$\left\{ \frac{\partial^2 L(r, \dot{r})}{\partial \dot{r}^i \partial \dot{r}^j} \right\}_{ij}$$

is positive definite for all $(r, \dot{r}) \in T\mathbb{R}^n$ so $L(r, \dot{r})$ is convex in $\dot{r}$.

This assumption is satisfied for most mechanical systems in view of the Riemannian form of the kinetic energy [17].

2.3. Hamiltonian Phase Spaces

The passage from the Lagrange formulation of the stationary conditions on $T\mathbb{G}$ to the Hamiltonian formulation on the cotangent bundle $T^*\mathbb{G}$, referred to as the phase space of the Hamiltonian system, is made by employing the Legendre transformation. In what follows, we apply the same methodology but using implicit representations. Again, since the system is constrained, its flow does not evolve on the whole $T^*\mathbb{R}^n$, so we devote this subsection to the characterization of the subbundle on which the flow evolves.

Finding the appropriate subbundle of $T^*\mathbb{R}^n$ is, surprisingly, more complicated than finding the appropriate subbundle of $T\mathbb{R}^n$. There are two difficulties: The inclusion $\imath$ does not define an injection from $T^*_q\mathbb{G}$ to $T^*_{\imath(q)}\mathbb{R}^n$, it rather defines a surjection (the pull-back) from $T^*_{\imath(q)}\mathbb{R}^n$ to $T^*_q\mathbb{G}$. Thus, the desired subbundle cannot be uniquely defined as there are many subsets of $T^*_{\imath(q)}\mathbb{R}^n$ with the same image $T^*_q\mathbb{G}$ under this map. Also, recall that there is no canonical isomorphism between a linear space and its first dual [15, p. 127], so the subspaces (1) do not define natural subspaces of $T^*_{\imath(q)}\mathbb{R}^n$. We will show, however, that by incorporating the knowledge of the system Lagrangian it is possible to construct a special submanifold $L\mathbb{G}$ of $T^*\mathbb{R}^n$ that accounts for the system constraints. An important part of this construction is played by the fiber derivative map, which also takes part of the Legendre transformation, as defined below [7].

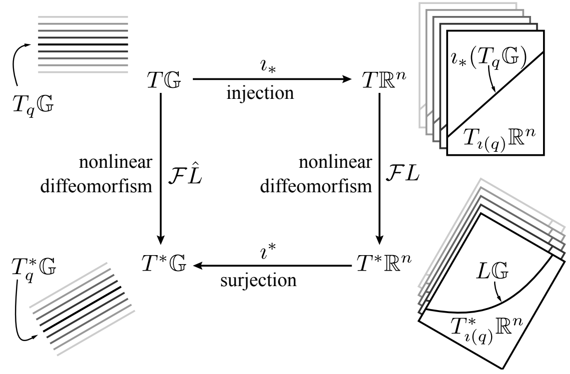

Figure 1: Commutative diagram showing the relationship between the phase spaces $T^*\mathbb{G}$ and $L\mathbb{G}$. Notice that $L\mathbb{G}$ is not a fiber bundle if $\mathcal{F}L$ is nonlinear (i.e., if L is not quadratic in $\dot{r}$).

Definition 4. *The fiber derivative of L is a map $\mathcal{F}L : T\mathbb{R}^n \to T^*\mathbb{R}^n$ defined by*

$$\langle \mathcal{F}L(X), Y\rangle = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} L(r, X + sY)$$

for all $X, Y \in T_r\mathbb{R}^n$ and any $r \in \mathbb{R}^n$, where $\langle\cdot,\cdot\rangle$ is the standard pairing of a vector and a covector, i.e. the value of a covector $\mathcal{F}L(X)$ as it acts on the vector Y.

The value $\langle \mathcal{F}L(X), Y\rangle$ is in fact equal to the variation in L at X along the fiber $T_r\mathbb{R}^n$ in the direction of Y. It is also easily verified that $\mathcal{F}L$ is fiber-preserving in the sense that it maps the fiber $T_r\mathbb{R}^n$ into the fiber $T_r^*\mathbb{R}^n$. Hereafter, the notation $\{r^i, \dot{r}^i\}$ and $\{r^i, p_i\}$ will be used to denote global coordinates on $T\mathbb{R}^n$ and $T^*\mathbb{R}^n$, respectively, while $\{q^i, \dot{q}^i\}$ and $\{q^i, \hat{p}_i\}$ denote local coordinates on $T\mathbb{G}$ and $T^*\mathbb{G}$. In charts, the fiber derivative takes the familiar form, $\mathcal{F}L : (r, \dot{r}) \mapsto (r, p) : p_i = \partial L(r, \dot{r})/\partial \dot{r}^i$ (i.e., it is a gradient mapping). We propose the following definition.

Definition 5. *The Legendre manifold $L\mathbb{G}$ is defined as the submanifold of $T^*\mathbb{R}^n$ given by $L_q\mathbb{G} := \mathcal{F}L \circ \imath_*(T_q\mathbb{G})$ and $L\mathbb{G} = \coprod_{q\in\mathbb{G}} L_q\mathbb{G} \subset T^*\mathbb{R}^n$.*

Note that, by virtue of Assumption 3, the fiber map $\mathcal{F}L$ is a $\mathcal{C}^{s-1}$-diffeomorphism of $\imath_*(T_q\mathbb{G})$ onto $L\mathbb{G}$; see [18, Lem. 1]. This fact allows us to define the desired subbundle using the defining functions and the Lagrangian alone, as illustrated in Fig. 1.

Proposition 6. *Let $\mathcal{F}\hat{L}$ denote the fiber derivative of the Lagrangian function $\hat{L} : T\mathbb{G} \to \mathbb{R}$, $\hat{L} = L \circ \varphi$, expressed in local coordinates on $T\mathbb{G}$. Let $\imath^*$ denote the pull-back of $\imath$. Then, the diagram of Fig. 1 commutes. Moreover, $\mathcal{F}\hat{L}$ is a $\mathcal{C}^{s-1}$-diffeomorphism of $T\mathbb{G}$ onto $T^*\mathbb{G}$.*

Proof. First it will be shown that $\mathcal{F}\hat{L} = \imath^* \circ \mathcal{F}L \circ \imath_*$. Applying Definition 4 to $\hat{L}$ gives $\langle \mathcal{F}\hat{L}(\xi), \eta\rangle = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \hat{L}(q, \xi + s\eta)$, where $\xi, \eta \in T_q\mathbb{G}$. Using $\hat{L} = L \circ \varphi$ in the right hand side $\langle \mathcal{F}\hat{L}(\xi), \eta\rangle = \langle \mathcal{F}L \circ \imath_*(\xi), \imath_*(\eta)\rangle$, for $\xi, \eta \in T_q\mathbb{G}$. Employing the definition of the pull-back map, this pairing can be written as $\langle \mathcal{F}\hat{L}(\xi), \eta\rangle = \langle \imath^* \circ \mathcal{F}L \circ \imath_*(\xi), \eta\rangle$, which must hold for all ξ and η, so $\mathcal{F}\hat{L} = \imath^* \circ \mathcal{F}L \circ \imath_*$.

In view of the smoothness of $\imath_*$, the restricted mapping $\hat{L}|_{T_q\mathbb{G}}$ is of class $\mathcal{C}^s(T_q\mathbb{G})$. To ascertain that $\mathcal{F}\hat{L}$ is a diffeomorphism, it must be shown that the Hessian of $\hat{L}$ is positive definite [18, Lem. 1]. Applying the chain rule to $L \circ \varphi = L \circ (\imath, \imath_*)$ gives

$$\frac{\partial \hat{L}}{\partial \dot{q}^j} = \frac{\partial \imath_*^l}{\partial \dot{q}^j}\frac{\partial L}{\partial \dot{r}^l} \circ (\imath, \imath_*),$$

where Einstein's summation convention was used to simplify notation. Application of the same rule yields the second derivatives,

$$\frac{\partial^2 \hat{L}}{\partial \dot{q}^i \partial \dot{q}^j} = \frac{\partial \imath_*^k}{\partial \dot{q}^i}\frac{\partial \imath_*^l}{\partial \dot{q}^j}\frac{\partial^2 L}{\partial \dot{r}^k \partial \dot{r}^l} \circ (\imath, \imath_*).$$

Since the Jacobian $\left\{\partial \imath_*^l/\partial \dot{q}^j\right\}_{lj}$ is full-rank, the Hessian is positive definite for all $(q, \dot{q})$ belonging to the same coordinate neighborhood. Each fiber $T_q\mathbb{G}$ can be covered using a single coordinate chart and $\mathcal{F}\hat{L}$ is fiber preserving, so $\mathcal{F}\hat{L}$ is a $\mathcal{C}^{s-1}$-diffeomorphism of $T\mathbb{G}$ onto $T^*\mathbb{G}$. $\square$

2.4. The Legendre transform

Once the invertibility of $\mathcal{F}L$ is insured by the validity of Assumption 3, it is correct to define the Hamiltonian function $H : T^*\mathbb{R}^n \to \mathbb{R}$ as the Legendre transformation of L [18, 19], with L viewed as function of $\dot{r}$ only, i.e.,

$$H(r, p) = \left(\dot{r}^i p_i - L(r, \dot{r})\right)\big|_{\dot{r} = \mathcal{F}L^{-1}(p)}. \tag{2}$$

The function H is equal to the total energy of the system, expressed using the redundant coordinates $\{r^i, p_i\}$. An interesting property of the Legendre transform is that it is its own inverse if L is convex [18, 19]. More precisely, the fiber derivative of H, $\mathcal{F}H : T^*\mathbb{R}^n \to T\mathbb{R}^n$, defined as

$$\langle W, \mathcal{F}H(Z)\rangle = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} H(r, Z + sW)\ ,$$

where $Z, W \in T_r^*\mathbb{R}^n$, is the inverse of $\mathcal{F}L$. In coordinates, $\mathcal{F}H : (r, p) \mapsto (r, \dot{r}) : \dot{r}^i = \partial H(r, p)/\partial p_i$. Also, the Lagrangian can be recovered by setting $L(r, \dot{r}) = \left(\dot{r}^i p_i - H(r, p)\right)\big|_{p = \mathcal{F}H^{-1}(\dot{r})}$. Likewise, the Hamiltonian function $\hat{H} : T^*\mathbb{G} \to \mathbb{R}$ is defined as

$$\hat{H}(q, \hat{p}) = \left(\dot{q}^i \hat{p}_i - \hat{L}(q, \dot{q})\right)\Big|_{\dot{q} = \mathcal{F}\hat{L}^{-1}(\hat{p})}. \tag{3}$$

The invertibility of $\mathcal{F}L$, together with Definition 5 and (1), implies that $L_q\mathbb{G} = \left\{Z \in T^*_{\imath(q)}\mathbb{R}^n \mid \mathcal{F}L^{-1}(Z)(g^i) = 0\right\}$. Since $\mathcal{F}H$ is the inverse of $\mathcal{F}L$, in local coordinates,

$$L_q\mathbb{G} = \left\{p \in T^*_{\imath(q)}\mathbb{R}^n \mid \frac{\partial H(r, p)}{\partial p_j}\frac{\partial g^i(r)}{\partial r^j} = 0\right\}. \tag{4}$$

The commutative diagram of Fig. 1 suggests that $\jmath^* := \mathcal{F}L \circ i_* \circ \mathcal{F}\hat{L}^{-1}$ is a right inverse for $\imath^*$. Indeed, $\imath^* \circ \jmath^* = \imath^* \circ \mathcal{F}L \circ \imath_* \circ \mathcal{F}\hat{L}^{-1} = \mathcal{F}\hat{L} \circ \mathcal{F}\hat{L}^{-1} = \mathrm{Id}$. Additionally, $\jmath^*(T^*\mathbb{G}) = L\mathbb{G}$. The Hamiltonian functions in global and local coordinates are then related as follows.

Proposition 7. *The Hamiltonians in implicit and explicit system representations are related by* $\hat{H} = H \circ (\imath, \jmath^*)$.

Thus, $\hat{H}$ corresponds to the total energy as well, but expressed using the local coordinates $\{q^i, \hat{p}_i\}$. This is a consequence of the fact that our map $(\imath, \jmath^*) : (q, \hat{p}) \mapsto (r, p)$ commutes with the Legendre transform.

Proof. It follows from (2) that

$$H \circ (\imath(q), \jmath^*(\hat{p})) = \left(\dot{r}^i \jmath_i^*(\hat{p}) - L(\imath(q), \dot{r})\right)\big|_{\dot{r}=\mathcal{F}L^{-1}\circ\jmath^*(\hat{p})} .$$

Since $\mathcal{F}L^{-1} \circ \jmath^* = \imath_* \circ \mathcal{F}\hat{L}^{-1}$ and $\dot{r} = \imath_*(\dot{q})$,

$$H \circ (\imath(q), \jmath^*(\hat{p})) = \left(\imath_*^i(\dot{q})\jmath_i^*(\hat{p}) - L(\imath(q), \imath_*(\dot{q}))\right)\big|_{\dot{q}=\mathcal{F}\hat{L}^{-1}(\hat{p})} .$$

Finally, the Hamiltonian (3) is recovered by recalling that $\hat{L} = L \circ (\imath, \imath_*)$ and noting that $\imath_*^i(\dot{q})\jmath_i^*(\hat{p}) = \dot{q}^i\hat{p}_i$. □

2.5. Hamilton's equations

Consider again the local coordinates $\{q^i, \hat{p}_i\}$ on the manifold $T^*\mathbb{G}$. A system is named *Hamiltonian* if its trajectories are integral curves of the Hamiltonian vector field $D_{\hat{H}} : T^*\mathbb{G} \to T(T^*\mathbb{G})$, where $D_{\hat{H}} = \frac{\partial\hat{H}}{\partial\hat{p}_i}\frac{\partial}{\partial q^i} - \frac{\partial\hat{H}}{\partial q^i}\frac{\partial}{\partial\hat{p}_i}$.

The Hamiltonian vector field unfolds into a more familiar ODE system:

$$\dot{q} = +\nabla_{\hat{p}}\,\hat{H}(q, \hat{p}) \ , \quad \dot{\hat{p}} = -\nabla_q\,\hat{H}(q, \hat{p}) \ . \tag{5}$$

See [6] for more details and a coordinate-free definition of the Hamiltonian vector field.

The implicit model for Hamiltonian systems is defined as follows. In global coordinates $\{r^i, p_i\}$ on $T^*\mathbb{R}^n$, the *implicit* Hamiltonian vector field $X_{H,g} : L\mathbb{G} \to T(L\mathbb{G})$, takes the form,

$$X_{H,g} = D_H - \lambda_j\frac{\partial g^j}{\partial r^i}\frac{\partial}{\partial p_i} \ , \quad g = 0 \ , \tag{6}$$

with $D_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial r^i} - \frac{\partial H}{\partial r^i}\frac{\partial}{\partial p_i}$. See [12] for details on the derivation of this expression. The Lagrange multipliers λ_j are defined implicitly by (6) and the restriction that the integral curve must lie on $L\mathbb{G}$. More concretely, by applying $X_{H,g}$ to both sides of the constraint equations $g^j = 0$, one obtains the *hidden constraints* $f^j := X_{H,g}(g^j) = \frac{\partial H}{\partial p_i}\frac{\partial g^j}{\partial r^i} = 0$. Application of $X_{H,g}$ to the hidden constraints makes the λ_j appear,

$$X_{H,g}(f^l) = D_H(f^l) - \lambda_j\frac{\partial g^j}{\partial r^i}\frac{\partial f^l}{\partial p_i} = 0 \ . \tag{7}$$

Thus, if the matrix

$$\left\{\frac{\partial g^j}{\partial r^i}\frac{\partial f^l}{\partial p_i}\right\}_{jl} = \left\{\frac{\partial g^j}{\partial r^i}\frac{\partial^2 H}{\partial p_i\partial p_m}\frac{\partial g^l}{\partial r^m}\right\}_{jl} \tag{8}$$

is non-singular on $L\mathbb{G}$, then there are unique λ_j satisfying (7) and ensuring that the integral curve stays on $L\mathbb{G}$. In mechanical systems, λ is the covector of constraint forces that ensure that the constraints are being enforced.

Remark 8. *Since the Hessian of* L *is positive definite, the Hessian of* H *is positive definite as well [18] (convexity is preserved by the Legendre transform). The matrix* (8) *is thus positive definite (hence invertible) on account of Assumptions 1 and 3.*

Remark 9. *The statements* $(r, p) \in L\mathbb{G}$ *and* $g(r) = 0$, $f(r, p) = 0$ *are equivalent (cf.* (4)*).*

The implicit vector field develops into the semi-explicit DAE

$$\dot{r} = \nabla_p H(r, p) \ , \ \dot{p} = -\nabla_r H(r, p) - G(r)^\top\lambda \ , \ 0 = g(r) \ , \tag{9}$$

where $G(r)$ is the Jacobian of $g(r)$.

Example: A double planar pendulum

Consider the model of a double planar pendulum (Fig. 2) that comprises a pair of point masses m_a and m_b whose coordinate positions are $r^a = (r^{a_x}, r^{a_y})$ and $r^b = (r^{b_x}, r^{b_y})$, respectively. The massless bars are of fixed lengths l_a and l_b, which gives rise to the two holonomic constraints:

$$g^1(r) = \|r^a\|^2 - l_a^2 = 0 \ , \quad g^2(r) = \|r^\delta\|^2 - l_b^2 = 0 \ , \tag{10}$$

where $r := (r^a, r^b) \in \mathbb{R}^n$, $n = 4$, $k = 2$, $r^\delta := r^b - r^a$. The rank of the constraint Jacobian is full since

$$\operatorname{rank} G(r) = \operatorname{rank}\begin{pmatrix} r^{a_x} & r^{a_y} & 0 & 0 \\ -r^{\delta_x} & -r^{\delta_y} & r^{\delta_x} & r^{\delta_y} \end{pmatrix} = k \tag{11}$$

for all $r \in \mathbb{G}$. Therefore, 0 is a regular value of g and $\mathbb{G}$ is an embedded submanifold of $\mathbb{R}^4$. The kinetic and potential energies of the system are

$$K(\dot{r}) = \frac{1}{2}\dot{r}^\top M\dot{r}, \quad M := \begin{pmatrix} m_a\mathbf{I} & \mathbf{0} \\ \mathbf{0} & m_b\mathbf{I} \end{pmatrix}, \tag{12}$$

$$V(r) = \bar{g}(m_a r^{a_y} + m_b r^{b_y}) \ , \tag{13}$$

where $\mathbf{0}$ and $\mathbf{I}$ are the null and identity elements in $\mathbb{R}^{2\times 2}$ and where $\bar{g}$ is the acceleration of gravity. The Lagrangian is $L = K - V$. The momentum is $p_i = \partial L/\partial\dot{r}_i$, i.e., $p = M\dot{r}$. It follows from (12) and (13) that the total energy, $K + V$, is

$$H(r, p) = \frac{1}{2}p^\top M^{-1}p + \bar{g}(m_a r^{a_y} + m_a r^{a_y}) \ . \tag{14}$$

Substituting (10) and (14) in (9) gives

$$\dot{r}^a = m_a^{-1}p_a \ , \quad \dot{r}^b = m_b^{-1}p_b \ ,$$

$$\begin{pmatrix} \dot{p}_{a_x} \\ \dot{p}_{a_y} \\ \dot{p}_{b_x} \\ \dot{p}_{b_y} \end{pmatrix} = -\begin{pmatrix} 0 \\ \bar{g}m_a \\ 0 \\ \bar{g}m_b \end{pmatrix} - 2\begin{pmatrix} r^{a_x} & -r^{\delta_x} \\ r^{a_y} & -r^{\delta_y} \\ 0 & r^{\delta_x} \\ 0 & r^{\delta_y} \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} , \tag{15}$$

which, together with (10), constitutes a set of DAEs describing the motion of the double pendulum. The multipliers λ_1 and λ_2 are the magnitudes of the internal forces along the two bars.

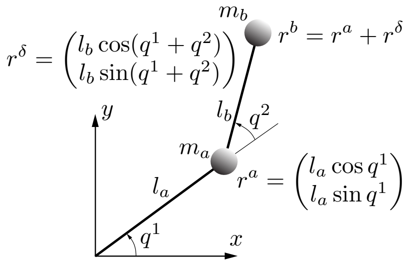

Figure 2: A double planar pendulum, a simple Hamiltonian system.

An explicit model for the double pendulum as an ODE is derived as follows. The dimension of $\mathbb{G}$ is $o = n - k = 2$. Motivated by Fig. 2, choose $q^1 \in (-\pi, \pi)$ and $q^2 \in (-\pi, \pi)$ as local coordinates for $\mathbb{G}$. Eq. (11) insures the existence of an embedding satisfying $g \circ \imath \equiv 0$. It can be readily verified that such embedding is

$$\begin{pmatrix} r^{a_x} \\ r^{a_y} \\ r^{b_x} \\ r^{b_y} \end{pmatrix} = \begin{pmatrix} l_a \cos q^1 \\ l_a \sin q^1 \\ l_a \cos q^1 + l_b \cos q^{\mathrm{t}} \\ l_a \sin q^1 + l_b \sin q^{\mathrm{t}} \end{pmatrix}, \quad q^{\mathrm{t}} := q^1 + q^2. \tag{16}$$

Direct differentiation of the two sides of (16) yields the following mapping between velocities,

$$\dot{r} = \begin{pmatrix} -l_a \sin q^1 & 0 \\ l_a \cos q^1 & 0 \\ -l_a \sin q^1 - l_b \sin q^{\mathrm{t}} & -l_b \sin q^{\mathrm{t}} \\ l_a \cos q^1 + l_b \cos q^{\mathrm{t}} & l_b \cos q^{\mathrm{t}} \end{pmatrix} \dot{q}\,, \quad q = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix}. \tag{17}$$

The expression for the kinetic energy in terms of the generalized positions and velocities, is obtained by substituting (17) into (12), with $K \circ \imath_*(\dot{q}) = \frac{1}{2}\dot{q}^\top \hat{M}(q)\dot{q}$,

$$\hat{M}(q) = \begin{pmatrix} m_{\mathrm{t}} l_a^2 + m_b l_b^2 + 2 m_b l_a l_b \cos q^2 & m_b l_b^2 + m_b l_a l_b \cos q^2 \\ m_b l_b^2 + m_b l_a l_b \cos q^2 & m_b l_b^2 \end{pmatrix}$$

and $m_{\mathrm{t}} = m_a + m_b$. The expression for the potential energy $V \circ \imath(q) = \bar{g} m_a l_a \sin q^1 + \bar{g} m_b (l_a \sin q^1 + l_b \sin q^{\mathrm{t}})$ is obtained by substituting (16) in (13). The vector of momenta is $\hat{p} = \hat{M}(q)\dot{q}$ and the total energy is

$$\hat{H}(q, \hat{p}) = \frac{1}{2}\hat{p}^\top \hat{M}(q)^{-1}\hat{p} + \bar{g}\left(m_{\mathrm{t}} l_a \sin q^1 + m_b l_b \sin q^{\mathrm{t}}\right). \tag{18}$$

Finally, Eq. (5) states that the motion of the system is described by

$$\dot{q} = \hat{M}(q)^{-1}\hat{p}\,,\ \dot{\hat{p}} = -\nabla_q V(q) - \nabla_q \left(\frac{1}{2}\hat{p}^\top \hat{M}(q)^{-1}\hat{p}\right). \tag{19}$$

Two representations for the same system were derived, one in the form of an ODE (19) and the other as a DAE (15). The main point of this example can be summarized in the following remark.

Remark 10. *For the DAE, the Hamiltonian function* (14) *is separable, i.e., the kinetic energy does not depend on* r*. The inertia matrix* M *is a* constant *diagonal matrix. Moreover, the potential energy* (13) *is* linear*, which results in a constant gradient. On the other hand, the Hamiltonian* (18) *that appears in the ODE is not separable (the inertia matrix depends on* q*) and it is composed of transcendental functions.*

The cost of a simpler Hamiltonian function is clear if one compares the dimensions of q and $\hat{p}$ that describe the system using the vector field (19) versus the dimensions of r and p in (15), and if one takes into account the need for computing the Lagrange multipliers. Despite this problems, implicit representations are particularly advantageous for the purposes of system discretization, accurate simulation and design of control laws.

2.6. Energy conservation and symplecticity

It is well known that the flow generated by $D_{\hat{H}}$ preserves the Hamiltonian function [10, 13]. In other words, Hamiltonian systems conserve energy, a property that is easily shown by computing $\mathcal{L}_{D_{\hat{H}}}\hat{H}$, the Lie derivative of $\hat{H}$ along the flow generated by $D_{\hat{H}}$. The implicit vector field (6), on the other hand, represents the same system as $D_{\hat{H}}$, so the flow generated by it must surely preserve the Hamiltonian function too [12]. Indeed,

$$\mathcal{L}_{X_{H,g}} H = X_{H,g}(H) = D_H(H) - \lambda_j \frac{\partial g^j}{\partial r^i}\frac{\partial H}{\partial p_i} = \lambda_j f^j. \tag{20}$$

Eq. (20) shows that $(\mathcal{L}_{X_{H,g}} H)(r, p) = 0$ for all $(r, p) \in L\mathbb{G}$ (cf. Remark 9), so H remains constant along the system trajectories.

It is also well known that, besides the 0-form $\hat{H}$, Hamiltonian flows preserve the 2-form $\hat{\omega} := \mathrm{d}q^i \wedge \mathrm{d}\hat{p}_i$, which acts on vectors of $T(T^*\mathbb{G})$ (See [6] for a coordinate-free definition). The invariance of $\hat{\omega}$ with respect to $D_{\hat{H}}$ can be established by showing that the Lie derivative $\mathcal{L}_{D_{\hat{H}}}\hat{\omega}$ is equal to zero, the demonstration being similar to that of the conservation of $\hat{H}$.

Definition 11. *A differentiable mapping* $\hat{\phi} : T^*\mathbb{G} \to T^*\mathbb{G}$ *is called* symplectic *if* $\hat{\phi}^*\hat{\omega} = \hat{\omega}$.

Employing the definition of the pull-back map, Definition 11 can be alternatively written as $\hat{\omega}(\hat{\phi}_*\xi, \hat{\phi}_*\eta) = \hat{\omega}(\xi, \eta)$ for all $\xi, \eta \in T_x(T^*\mathbb{G})$. Remarkably, the flow $\hat{\phi}_t$ that is generated by the Hamiltonian vector field $D_{\hat{H}}$ is symplectic.

Again, since the implicit Hamiltonian vector field $X_{H,g}$ refers to the same physical system as $D_{\hat{H}}$, one might reasonably expect $X_{H,g}$ to generate a symplectic flow. Similarly to $\hat{\omega}$, we define $\omega := \mathrm{d}r^i \wedge \mathrm{d}p_i$, which acts on vectors of $T(T^*\mathbb{R}^n)$.

Theorem 12. *[12, 14] Let* H *be twice continuously differentiable. The flow* $\phi_t : L\mathbb{G} \to L\mathbb{G}$ *of* $X_{H,g}$ (6) *is a symplectic transformation on* $L\mathbb{G}$*, i.e.,* $\phi_t^*\omega = \omega$ *for every* t *for which* ϕ_{t} *is defined.*

Remark 13. *The converse statement, that* every symplectic flow ϕ_t solves Hamilton's equations for some H, *is also true, so symplecticity is a characteristic property of Hamiltonian systems [6]. This does not translate to the case of energy conservation, i.e., while every Hamiltonian system conserves energy, not every energy-conserving system is Hamiltonian.*

3. Adding ports

In the presence of external forces and dissipation it is convenient to represent $D_{\hat{H}}$ as an input–output system equipped with a pair of port variables (u, y), giving rise to a *PH system* [10, 13, 20] that is described by the vector field, $X_{\hat{H},u} : T^*\mathbb{G} \times (\mathbb{R}^m)^* \to T(T^*\mathbb{G})$,

$$X_{\hat{H},u} = D_{\hat{H}} + u_l \hat{U}_i{}^l \frac{\partial}{\partial \hat{p}_i} \, . \tag{21}$$

Here, $u \in (\mathbb{R}^m)^*$ is the controlled or input variable. The dependent or output variable, $y \in \mathbb{R}^m$, is such that $y^l = \hat{U}_i{}^l \frac{\partial \hat{H}}{\partial \hat{p}_i}$, where $\hat{U}_i{}^l$ are maps from $\mathbb{G}$ to $\mathbb{R}$. Applying this idea to the implicit Hamiltonian system (6) gives the implicit control vector field $X_{H,u,g} : L\mathbb{G} \times (\mathbb{R}^m)^* \to T(L\mathbb{G})$,

$$X_{H,u,g} = D_H + \left(u_l U_i{}^l - \lambda_j \frac{\partial g^j}{\partial r^i} \right) \frac{\partial}{\partial p_i} \, , \quad g = 0 \, , \tag{22}$$

with the output defined by $y^l = U_i{}^l \frac{\partial H}{\partial p_i}$. A system described by (22) is called an *implicit port-Hamiltonian system* [10]

The vector field (22) and the output unravel to become,

$$\dot{r} = +\nabla_p H(r,p) \, , \; \dot{p} = -\nabla_r H(r,p) - G(r)^\top \lambda + U(r) u \, ,$$
$$y = U(r)^\top \nabla_p H(r,p) \, , \; 0 = g(r) \, .$$

By analogy with the results described in Sec. 2.5, one can determine the Lagrange multipliers λ explicitly. The constraints $f^a = 0$ imply that

$$X_{H,u,g}(f^a) = D_H(f^a) + u_l U_i{}^l \frac{\partial f^a}{\partial p_i} - \lambda_j \frac{\partial g^j}{\partial r^i} \frac{\partial f^a}{\partial p_i} = 0 \, , \tag{23}$$

from which it follows that, as long as (8) is non-singular, there are unique λ_j (in general dependent on u as well as on r and p) such that $X_{H,u,g}(f^a) = 0$ and such that the integral curve stays on $L\mathbb{G}$.

It follows from Proposition 7 and $y^l = \hat{U}_i{}^l \frac{\partial \hat{H}}{\partial \hat{p}_i} = U_i{}^l \frac{\partial H}{\partial p_i}$ that

$$\hat{U}_i{}^l = \imath_*{}^j{}_i \cdot (U \circ \imath)_j{}^l \, . \tag{24}$$

3.1. Passivity

It can be readily seen that an implicit PH system described by (22) no longer preserves H. The Lie derivative of H is now $\mathcal{L}_{X_{H,u,g}}(H) = u_l y^l - \lambda_j f^j$. Recall that $f^j = 0$ for all j and all $(r,p) \in L\mathbb{G}$, so one has the power balance $\mathcal{L}_{X_{H,u,g}} H = u_l y^l$. Since the product $u_l y^l$ is equal to the rate of change in energy, we say that (u, y) is a *power-conjugated pair of port variables.* If, in addition, the restriction of H to $L\mathbb{G}$ is bounded from below, i.e., if the image of $L\mathbb{G}$ under H is bounded from below, then (22) is called passive, or more precisely, *lossless.* Boundedness of H can be easily assessed using the following proposition, which is a consequence of Weierstrass' Theorem.

Proposition 14. *Suppose that the potential energy V is lower semi-continuous and $\mathbb{G}$ is compact in the topology of $\mathbb{R}^n$. Then, $H(L\mathbb{G})$ is bounded from below (hence, the vector field* (22) *describes a lossless system).*

Example: A double planar pendulum (continued)

Suppose that the double pendulum (Fig. 2) is actuated by application of torques u_1 and u_2 to the joints that correspond to the angles q^1 and q^2, respectively. The resulting linear forces $U^1 u_1$ and $U^2 u_2$ ($U^1 := \{U_i{}^1\}_i$ and $U^2 := \{U_i{}^2\}_i$) can be computed using (24), that is,

$$U^1 = \begin{pmatrix} -r^{a_y} \\ r^{a_x} \\ 0 \\ 0 \end{pmatrix} \frac{1}{l_a^2} \quad \text{and} \quad U^2 = \begin{pmatrix} r^{\delta_y} \\ -r^{\delta_x} \\ -r^{\delta_y} \\ r^{\delta_x} \end{pmatrix} \frac{1}{l_b^2} - U^1 \, .$$

The manifold defined by (10) is compact and the potential energy (13) is continuous, which confirms that the double pendulum is passive with passive outputs $y^l = U_i{}^l \partial H / \partial p_i = U_i{}^l \dot{r}^i$. From (16), (17) and (24), we have $\hat{U}_1{}^1 = 1$, $\hat{U}_2{}^1 = 0$, $\hat{U}_1{}^2 = 0$ and $\hat{U}_2{}^2 = 1$, so that in local coordinates the passive outputs correspond to the angular velocities $y^1 = \hat{U}_i{}^1 \dot{q}^i = \dot{q}^1$ and $y^2 = \hat{U}_i{}^2 \dot{q}^i = \dot{q}^2$.

3.2. The evolution of the symplectic form

With the inclusion of the control variable u, it can no longer be expected that the flow of (22) be symplectic.

Proposition 15. *The Lie derivative of $\omega = \mathrm{d}r^i \wedge \mathrm{d}p_i$ restricted to $L\mathbb{G}$ satisfies*

$$\mathcal{L}_{X_{H,u,g}} \omega \big|_{L\mathbb{G}} = \mathrm{d}r^i \wedge \mathrm{d}(u_l U_i{}^l) \, . \tag{25}$$

Proof. Recall first that the Lie derivative of a general l-form α along a vector field X can be computed using Cartan's formula [21] as $\mathcal{L}_X \alpha = \mathrm{d}\,(i_X \alpha) + i_X \mathrm{d}\alpha$. That is, $\mathcal{L}_X \alpha$ is the sum of two l-forms: $\mathrm{d}\,(i_X \alpha)$, the exterior derivative of $i_X \alpha$ (the contraction of α on X), and the contraction of $\mathrm{d}\alpha$ on X. Thus,

$$\begin{aligned} \mathcal{L}_{X_{H,u,g}} \omega &= \mathrm{d}\left(i_{X_{H,u,g}} (\mathrm{d}r^i \wedge \mathrm{d}p_i) \right) + i_{X_{H,u,g}} \mathrm{d}(\mathrm{d}r^i \wedge \mathrm{d}p_i) \\ &= \mathrm{d}\Big((i_{X_{H,u,g}} \mathrm{d}r^i) \wedge \mathrm{d}p_i - \mathrm{d}r^i \wedge (i_{X_{H,u,g}} \mathrm{d}p_i) \Big) \, . \end{aligned}$$

The second equality follows from the anti-derivation property of the contraction and the wedge product [21, p. 152] and the fact that $\mathrm{d}^2 \beta = 0$ for any differential form β. Performing the contraction of $\mathrm{d}r^i$ and $\mathrm{d}p_i$ on $X_{H,u,g}$ gives,

$$\mathcal{L}_{X_{H,u,g}} \omega = \mathrm{d}\left(\frac{\partial H}{\partial p_i} \mathrm{d}p_i + \mathrm{d}r^i \left(\frac{\partial H}{\partial r^i} - u_l U_i{}^l + \lambda_j \frac{\partial g^j}{\partial r^i} \right) \right)$$

	Explicit model	Implicit model	Corresponding map
Positions	q	r	$r = \imath(q)$
Velocities	$\dot{q}$	$\dot{r}$	$\dot{r} = \imath_*(\dot{q})$
Momenta	$\hat{p}$	p	$p = \jmath^*(\hat{p})$
Control vector fields	$\hat{U}_i{}^l\,\partial/\partial\hat{p}_i$	$U_i{}^l\partial/\partial p_i$	$\hat{U}_i{}^l(q) = \imath_*{}^j{}_i(q)\cdot U_j{}^l(\imath(q))$
Lagrangian	$\hat{L}(q,\dot{q})$	$L(r,\dot{r})$	$\hat{L}(q,\dot{q}) = L(\imath(q),\imath_*(q))$
Hamiltonian	$\hat{H}(q,\hat{p})$	$H(r,p)$	$\hat{H}(q,\hat{p}) = H(\imath(q),\jmath^*(\hat{p}))$
Symplectic form	$\hat{\omega} = \mathrm{d}q^i\wedge\mathrm{d}\hat{p}_i$	$\omega = \mathrm{d}r^i\wedge\mathrm{d}p_i$	

Table 1: Relationship between the implicit and explicit representations of a PH system.

or

$$\mathcal{L}_{X_{H,u,g}}\omega = \mathrm{d}r^i\wedge\mathrm{d}(u_l U_i{}^l) + \mathrm{d}\lambda_j\wedge\mathrm{d}g^j \qquad (26)$$

(see [21, Ch. 11] or [14] for a list of rules for the exterior derivative, contraction and wedge product). The constraint $g=0$ implies the restrictions $\mathrm{d}g^j = 0$, i.e., $\mathrm{d}g^j(\xi)=0$ for all $\xi\in T_x(L\mathbb{G})$, so (26) reduces to (25). □

Remark 16. *Equation* (25) *shows that, when* $u\equiv 0$*, the Lie derivative of* ω *is equal to zero, so* ω *remains constant and in consequence the flow is symplectic.*

There is another particular case in which the flow can be made symplectic: Suppose that u_l are functions on $\mathbb{G}$ such that $u_l U_i{}^l = \partial V_\mathrm{a}/\partial r^i$ for some artificial potential function $V_\mathrm{a}:\mathbb{G}\to\mathbb{R}$. Then, $\mathcal{L}_{X_{H,u,g}}\omega\big|_{L\mathbb{G}} = \mathrm{d}r^i\wedge\mathrm{d}(u_l U_i{}^l) = \mathrm{d}r^i\wedge\frac{\partial^2 V_\mathrm{a}}{\partial r^j\partial r^i}\mathrm{d}r^j = 0$. Equation $u_l U_i{}^l = \partial V_\mathrm{a}/\partial r^i$ appears, e.g., in the context of passivity based control, when performing potential energy shaping [22]. In general, however, u destroys symplecticity.

4. Conclusions and Future Work

The complete geometric relation between implicit and explicit port-Hamiltonian systems is summarized in Table 1. This material can be useful when comparing or developing different integration or controller design methods. It is also possible to design a controller using one representation and to implement it using the other.

As a starting point, we have applied a modified (implicit) version of energy shaping PBC in order to stabilize the upward equilibrium of the double pendulum. The control problem is trivial but it is interesting to note that, in the implicit framework, simple algebraic equations appear in place of the usual partial differential equations. When proper damping is applied, asymptotic stability can be proved using Proposition 15 instead of the usual LaSalle argument. Details will be reported elsewhere.

The authors are currently using this material to develop numerical integration schemes for Hamiltonian systems with inputs (the autonomous case is well developed, but the non autonomous is new).

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