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A set-valued nested sliding-mode controller

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Abstract: We propose a set-valued controller with a signum multifunction nested inside another one. We prove that the controller is well-posed and achieves robust ultimate boundedness in the presence of mismatched, non-vanishing disturbances. Even more, the selected output can be made arbitrarily small. Also, by applying the implicit Euler scheme introduced by Acary and Brogliato [2010], Acary et al. [2012] for matched disturbances, we derive a selection strategy for the discrete-time implementation of the set-valued control law. The discrete-time scheme diminishes chattering substantially.

Keywords: Differential inclusions, digital implementation, robust control, mismatched disturbances, linear uncertain systems, Lyapunov stability.

1. INTRODUCTION

Since its appearance, sliding-mode control has been studied in detail because of its robustness against parametric and matched disturbances [Utkin et al. 2009, Slotine and Sastry 1983, Levant 2003, Young et al. 1999]. However, this kind of controllers is known to be fragile in the presence of mismatched disturbances, that is, disturbances that affect the system through a channel not shared by the control input, and which in consequence cannot be eliminated by applying the conventional methods [Utkin et al. 2009].

A distinctive feature of first-order sliding-mode control is the finite-time convergence of the state towards the so-called sliding surface. However, if the sliding surface is linear, the state will approach the origin only exponentially fast, even while in the sliding regime. To achieve finite-time convergence to the origin, terminal sliding-mode control was proposed [Venkataraman and Gulati 1993, Zhihong et al. 1994]. For a system of the form (4) below, without mismatched perturbations, i.e., when $w_1(t,x) \equiv 0$, one defines a sliding variable $s(x) = x_2 + |x_1|^p \operatorname{sgn} x_1$ with $0 < p < 1$. It is not difficult to verify that, along the sliding surface $\{x \in \mathbb{R}^2 \mid s(x) = 0\}$, the state converges indeed to the origin in finite time. In this paper we study the limiting case $p = 0$, which results in a controller with a signum function nested inside another one. The controlled system is studied under a differential inclusions perspective.

The paper by Adhami-Mirhosseini and Yazdanpanah [2005] is also closely related to our work. The authors consider the case $p = 0$, but with a sigmoidal approximation in place of the true signum multifunction. In this regard, our work can also be understood as a limiting case of the controller proposed by Adhami-Mirhosseini and Yazdanpanah [2005], which is designed from a backstepping algorithm.

The main results are presented in two stages. First, we prove the well-posedness (existence of solutions) of the nested sliding-mode control algorithm by using a set-valued framework. The main result in this part is the ultimate boundedness of the closed-loop system in the presence of mismatched disturbances. In the second stage we present a methodology for the selection of the values of the control law which significantly alleviates the chattering effect by using an implicit Euler discretization in combination with a backstepping-like algorithm.

The paper is organized as follows: Section 2 recalls some results of stability theory in the nonsmooth setting. Section 3 is dedicated to the well-posedness as well as stability issues of the closed-loop system in continuous-time, whereas Section 4 studies the discrete-time counterpart of the nested controller. In Section 5 we present numerical results and the comparison of the implicit discretization against explicit techniques. Finally, the paper ends with the conclusions and possible future work.

2. PRELIMINARIES

Along all this paper we deal with set-valued maps, that is, maps that take a subset of the range for each point in their domain. Let $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map, the graph of $\mathbf{F}$ is given as $\operatorname{Graph} \mathbf{F} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n | y \in \mathbf{F}(x)\}$. Let $\mathcal{S}_1 \subset \mathbb{R}^n$ and $\mathcal{S}_2 \subset \mathbb{R}^n$ be two sets. The sum $\mathcal{S}_1 + \mathcal{S}_2$ is the set $\{\sigma \in \mathbb{R}^n | \sigma = \sigma_1 + \sigma_2,\ \sigma_1 \in \mathcal{S}_1,\ \sigma_2 \in \mathcal{S}_2\}$. The set-valued function $\operatorname{Sgn} : \mathbb{R} \rightrightarrows \mathbb{R}$ is given by

$$\operatorname{Sgn}(x) := \begin{cases} \operatorname{sgn}(x) & \text{if } x \neq 0 \\ [-1,1] & \text{otherwise} \end{cases},$$

whereas the single-valued function $\operatorname{sgn} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is defined as

$$\operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

Note that this single-valued signum function is undefined at $x = 0$.

Definition 1. A set-valued map $\mathbf{F} : X \rightrightarrows Y$ is called upper semi-continuous (usc) at $x_0 \in X$ if, for any open neighborhood M of $\mathbf{F}(x_0)$, there exists a neighborhood N of x_0 such that $\mathbf{F}(N) \subset M$. The set-valued map $\mathbf{F}$ is upper semi-continuous if it is so at every $x_0 \in X$.

Proposition 2. [Aubin and Cellina 1984] Let $\mathbf{F} : X \rightrightarrows Y$ and $\mathbf{G} : Z \rightrightarrows X$ be two set-valued usc maps. Then, the composition map $\mathbf{F} \circ \mathbf{G} : Z \rightrightarrows Y$ such that

$$\mathbf{F} \circ \mathbf{G}(x) := \cup_{y \in \mathbf{G}(x)} \mathbf{F}(y). \tag{1}$$

is also usc.

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function and consider the following differential inclusion,

$$\dot{x} \in \mathbf{F}(x), \quad x(0) = x_0, \tag{2}$$

where $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued upper semi-continuous map. The set-valued derivative of V along the trajectories of (2) is defined as [Bacciotti and Ceragioli 1999]

$$\mathcal{L}_{\mathbf{F}} V(x) := \{a \in \mathbb{R} \mid \exists\, v \in \mathbf{F}(x) \text{ such that}, \\ \langle p, v \rangle = a, \text{ for all } p \in \partial V(x)\}, \tag{3}$$

where the term $\partial V(x)$ refers to Clarke's subdifferential of the function V at the point x (see e.g., Clarke et al. [1998], Theorem 8.1, p. 93).

Theorem 3. [Bacciotti and Ceragioli 1999, Theorem 2] Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a locally Lipschitz, regular (in Clarke's sense, see, e.g., Clarke et al. [1998], Section 2.4) and positive definite function such that for all $x \in \mathbb{R}^n$, $\max\{\mathcal{L}_{\mathbf{F}} V(x)\} \le 0$. Then, the origin of $\dot{x} \in \mathbf{F}(x)$ is stable.

The following nonsmooth chain rule, adapted from Clarke [1990], Theorem 2.3.9, will be useful in the computation of Clarke's subdifferential

Proposition 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given as $f = g \circ h$, where $h : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz continuous function and $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then,

$$\partial f(x) = \nabla g(h(x)) \partial h(x).$$

3. CONTINUOUS-TIME NESTED SET-VALUED CONTROL

Consider the system

$$\begin{cases} \dot{x}_1 = x_2 + w_1(t, x_1) \\ \dot{x}_2 = u + w_2(t, x) \\ y = x_1 \end{cases}, \tag{4}$$

where $x_i \in \mathbb{R}$ represents the states of the system, $w_i \in \mathbb{R}$ accounts for external disturbances and unmodeled dynamics, $i = 1, 2$, and $u, y \in \mathbb{R}$ are the control input and desired output, respectively.

Along all this note we make the following assumption.

Assumption 5. The disturbance terms w_i, $i = 1, 2$, are locally Lipschitz continuous with respect to x and uniformly bounded in the $\mathcal{L}_\infty$ sense by positive constants W_i, that is, $\|w_i\|_{\mathcal{L}_\infty} \le W_i$ for all x.

Remark 6. Under the appropriate conditions on the system relative degree, the form (4) can be obtained by applying input–output linearization on a more general system

$$\begin{cases} \dot{z} = f(z) + g(z)v + p(z)\xi(t, z) \\ y = h(z) \end{cases}$$

with state $z \in \mathbb{R}^2$, control input $v \in \mathbb{R}$ and disturbance $\xi \in \mathbb{R}$, cf. Sastry [1999], (9.91) p. 418.

Objective: To regulate the output $y = x_1$ to a neighborhood of the origin in the presence of disturbances w_i, $i = 1, 2$.

Note that, even for this simple plant, the task of regulating the output is a challenging problem because of the presence the unmatched disturbance w_1. In order to achieve the robust regulation of y to a neighborhood of the origin, we propose the control law

$$u(x) \in -\gamma_2 \Xi(x) - \gamma_3 \operatorname{Sgn}(\Xi(x)), \tag{5}$$

where $\Xi : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ is the set valued map $x \mapsto \{x_2\} + \gamma_1 \operatorname{Sgn}(x_1)$ and the gains γ_i, $i = 1, 2, 3$, are positive and constant. The composed multifunction $\operatorname{Sgn}(x_2 + \gamma_1 \operatorname{Sgn}(x_1))$ is computed from (1) as

$$\operatorname{Sgn}(\Xi(x)) = \begin{cases} \operatorname{sgn}(\xi(x)) & \text{if } x_1 \neq 0,\ 0 \notin \Xi(x) \\ \operatorname{sgn}(x_2) & \text{if } x_1 = 0,\ 0 \notin \Xi(x) \\ [-1, 1] & \text{if } x_1 \in \mathbb{R},\ 0 \in \Xi(x) \end{cases},$$

where $\xi : \{x \in \mathbb{R}^2 \mid x_1 \neq 0\} \to \mathbb{R}$ is a singled-valued map given as $\xi(x) = x_2 + \gamma_1 \operatorname{sgn}(x_1)$. With the interconnection of (5) and (4), the closed-loop system becomes the differential inclusion

$$\begin{cases} \dot{x}_1 = x_2 + w_1 \\ \dot{x}_2 \in -\gamma_2 \Xi(x) - \gamma_3 \operatorname{Sgn}(\Xi(x)) + w_2 \\ y = x_1 \end{cases}. \tag{6}$$

The well-posedness of the closed-loop system (6) is now immediate in view of Smirnov [2002], Corollary 4.4, and Proposition 2 (and the fact that the sum of two usc operators is also usc), that is, for any initial condition $x = x_0$ there exists (at least) one absolutely continuous function $x : \mathbb{R}_+ \to \mathbb{R}^n$ such that (6) holds almost everywhere. However, uniqueness of solutions requires further properties which we do not investigate here, keeping in mind that the stability proofs in the sequel accommodate non-uniqueness of solutions.

It is also important to remark that, because of the presence of the persistent mismatched uncertain terms, it is impossible to drive the whole state to the origin. Thus, we aim for stability of a set rather than just a point.

3.1 Ultimate boundedness of the closed-loop

We start with some concepts from the theory of monotone operators that will be used along this section.

Definition 7. Let $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map. Then, $\mathbf{F}$ is monotone if for any $(x_1, y_1) \in \operatorname{Graph} \mathbf{F}$ and any $(x_2, y_2) \in \operatorname{Graph} \mathbf{F}$, $\langle y_2 - y_1, x_2 - x_1 \rangle \ge 0$. A monotone operator $\mathbf{F}$ is called maximal if its graph is not strictly contained in the graph of any other monotone operator.

There are two single-valued Lipschitz continuous maps associated with a maximal monotone operator: the map

$J^{\varepsilon}_{\mathbf{F}}$ called the resolvent of $\mathbf{F}$ and the Yosida approximation of $\mathbf{F}$ denoted as $\mathcal{F}^{\varepsilon}$. More specifically,

$$J^{\varepsilon}_{\mathbf{F}} := (I + \varepsilon \mathbf{F})^{-1} \tag{7}$$

$$\mathcal{F}^{\varepsilon} := \frac{1}{\varepsilon}\left(I - J^{\varepsilon}_{\mathbf{F}}\right), \tag{8}$$

where I is the identity operator. The following fact is taken from Aubin and Cellina [1984].

Proposition 8. Let $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a maximal monotone operator. Then, the Yosida approximation $\mathcal{F}^{\varepsilon}$ is a Lipschitz continuous map with constant $1/\varepsilon$, whereas the resolvent $J^{\varepsilon}_{\mathbf{F}}$ is non-expansive. Additionally, for all $x \in \mathbb{R}^n$, $\mathcal{F}^{\varepsilon}(x) \in \mathbf{F}(J^{\varepsilon}_{\mathbf{F}}(x))$.

From Definition 7 it is easily seen that the set-valued map Sgn is in fact maximal monotone. Let us define the set-valued maximal monotone operator $\mathbf{S} := \mathrm{Sgn}$. Then we have

$$J^{\varepsilon}_{\mathbf{S}}(x) = \begin{cases} x - \varepsilon\,\mathrm{sgn}(x) & \text{if } |x| \geq \varepsilon \\ 0 & \text{if } |x| < \varepsilon \end{cases} \tag{9}$$

$$\mathcal{S}^{\varepsilon}(x) = \begin{cases} \mathrm{sgn}(x) & \text{if } |x| \geq \varepsilon \\ \dfrac{x}{\varepsilon} & \text{if } |x| < \varepsilon \end{cases}. \tag{10}$$

Note that $\varepsilon \mathcal{S}^{\varepsilon}(x) = \mathrm{Proj}(x, [-\varepsilon, \varepsilon])$, where $\mathrm{Proj}(x, \mathcal{C})$ refers to the projection operator (see, e.g., [Hiriart-Urruty and Lemaréchal 1993, Section III.3]). It is worth remarking that, in general, neither the composition nor the sum of two set-valued maximal monotone maps is maximal monotone. In particular, the operator $\mathrm{Sgn}(x_2 + \gamma_1 \mathrm{Sgn}(x_1))$ is not maximal monotone.

Now we are ready to formulate the ultimate boundedness of solutions of the closed-loop system (6).

Theorem 9. Consider the closed-loop (6). Then, the set $\mathcal{Q} := [-\varepsilon, \varepsilon] \times [-\gamma_1, \gamma_1]$ is globally asymptotically stable, whenever

$$\gamma_1 > W_1, \tag{11a}$$

$$\gamma_2 > \frac{\gamma_1}{\varepsilon}, \tag{11b}$$

$$\gamma_3 \geq W_2 + \max\left\{1, \gamma_1\left(\gamma_2 + \frac{1}{\varepsilon}W_2\right)\right\}, \tag{11c}$$

where $\varepsilon > 0$ is fixed.

Proof. Consider the following nonsmooth, Lipschitz continuous, positive function:

$$V(x_1, x_2) = \mathrm{dist}(x_1, [-\varepsilon, \varepsilon]) + \frac{1}{2}\left(x_2 + \gamma_1 \mathcal{S}^{\varepsilon}(x_1)\right)^2, \tag{12}$$

where the function $\mathrm{dist}(\cdot, \mathcal{C}) : \mathbb{R}^n \to \mathbb{R}_+$ refers to the distance function from a point to a set $\mathcal{C}$. Next, we will use the function V for proving the stability of the set $[-\varepsilon, \varepsilon] \times [-\gamma_1, \gamma_1]$. Therefore, following the nonsmooth chain rule in Proposition 4 along with [Burke et al. 1992, Theorem 1], we obtain

$$\partial V(x) = \begin{cases} \left(\mathrm{sgn}(x_1), \xi(x)\right) & \text{if } |x_1| > \varepsilon \\ \left([0,1](\mathrm{sgn}(x_1) + \frac{\gamma_1}{\varepsilon}\xi(x)), \xi(x)\right) & \text{if } |x_1| = \varepsilon \\ \left(\frac{\gamma_1}{\varepsilon}\left(x_2 + \frac{\gamma_1}{\varepsilon}x_1\right), x_2 + \frac{\gamma_1}{\varepsilon}x_1\right) & \text{if } |x_1| < \varepsilon \end{cases}. \tag{13}$$

In order to compute the set-valued derivative of V along the trajectories of (6) we set

$$\mathbf{F}(x) := \begin{bmatrix} x_2 + w_1 \\ -\gamma_2 \Xi(x) - \gamma_3 \,\mathrm{Sgn}\left(\Xi(x)\right) + w_2 \end{bmatrix}.$$

We split the analysis in three cases. For space limitations we sketch only the first case, the remaining two are developed in a similar fashion.

Case 1 ($|x_1| > \varepsilon$). From (3) together with (13) we obtain

$$\mathcal{L}_{\mathbf{F}}V(x) = \{a \in \mathbb{R} \mid a = \mathrm{sgn}(x_1)\,(x_2 + w_1) + \xi(x)\,(-\gamma_2 \xi(x) - \gamma_3 \zeta + w_2)\,,\ \zeta \in \mathrm{Sgn}(\xi(x))\}\,.$$

Hence, it follows that

$$\max\{\mathcal{L}_{\mathbf{F}}V(x)\} \leq -(\gamma_1 - W_1) - \gamma_2 \xi(x)^2 - (\gamma_3 - W_2 - 1)|\xi(x)|$$

which is strictly negative whenever $\gamma_1 > W_1$, $\gamma_2 > 0$ and $\gamma_3 > 1 + W_2$.

By taking the cases where $|x_1| = \varepsilon$ and $|x_1| < \varepsilon$ with $|x_2| > \gamma_1$, after some standard computations we conclude that the set-valued derivative of V is strictly negative everywhere outside the rectangle $\mathcal{Q}$, which yields the global stability of $\mathcal{C}$ and the ultimate boundedness of the trajectories of (6).

4. DISCRETE-TIME NESTED SET-VALUED CONTROL

In this section we study the discrete-time counterpart of the nested controller introduced in Section 3. It has been shown in Acary and Brogliato [2010], Acary et al. [2012], Huber et al. [2016a,b], Miranda-Villatoro et al. [2017, 2016] that the adequate selection of the values of a set-valued controller can substantially reduce the chattering effect. The main contributions of the aforementioned works rely on an implicit discretization scheme which, by using a nominal model of the closed-loop, makes the selection of the values of the controller such that (matched) disturbances are compensated.

Roughly speaking, we will exploit the triangular structure structure (4) by applying a backstepping-like algorithm. The algorithms reported in Acary and Brogliato [2010], Huber et al. [2016a], Miranda-Villatoro et al. [2016] will be then used to yield a suitable discrete-time control law that exhibits drastically lower chattering than the 'conventional' explicit Euler discretization.

After applying an implicit Euler discretization to (4) we obtain the discrete-time version of the disturbed double-integrator,

$$\begin{cases} x_{1,k+1} = x_{1,k} + h\,(x_{2,k+1} + w_{1,k}) & (14a) \\ x_{2,k+1} = x_{2,k} + h(u_k + w_{2,k}) & (14b) \\ y_k = x_{1,k}, & (14c) \end{cases}$$

where $h = t_{k+1} - t_k > 0$ represents the sampling period (considered fixed), and the notation $f_{i,k}$ stands for $f_i(t_k)$. Now we go one step through the backstepping-like algorithm: We consider the virtual system

$$x_{1,k+1} = x_{1,k} + h\nu_k + hw_{1,k}$$

with virtual control input ν_k and we consider its nominal version

$$\tilde{x}_{1,k+1} = x_{1,k} + h\nu_k. \tag{15}$$

We know that the set-valued control law

$$\nu_k \in -\gamma_1 \,\mathrm{Sgn}(\tilde{x}_{1,k+1}) \tag{16}$$

is well-posed, that is, it allows for a suitable selection strategy, achieves the robust regulation of the virtual state

$\tilde{x}_{1,k}$ to the origin in finite time for $\gamma_1 > 0$ sufficiently large and admits the Lyapunov function $V_1 = |x_{1,k}|$ (see, e.g., Acary and Brogliato [2010], Huber et al. [2016a], Miranda-Villatoro et al. [2016]). It is worth recalling that steering $\tilde{x}_{1,k}$ towards the origin implies that $|x_{1,k}| \leq hW_1$. Then we have the limit $x_{1,k} \to 0$ as $h \to 0$.

Next, we go one step further into the backstepping algorithm by considering the whole system (14) and the control law

$$u_k = -\gamma_2(x_{2,k} - \nu_k) + \eta_k, \tag{17}$$

where ν_k is as in (15)-(16) and

$$\eta_k \in -\gamma_3 \operatorname{Sgn}(\tilde{x}_{2,k+1} - \nu_k) \tag{18}$$
$$\tilde{x}_{2,k+1} = x_{2,k} + hu_k. \tag{19}$$

We see that u_k in (17) is a particular discretization of $u(x)$ in (5) with mixed explicit and implicit terms as:

$$u_k \in -\gamma_2 (x_{2,k} + \gamma_1 \operatorname{Sgn}(\tilde{x}_{1,k+1})) - \gamma_3 \operatorname{Sgn}(\tilde{x}_{2,k+1} + \gamma_1 \operatorname{Sgn}(\tilde{x}_{1,k+1})),$$

where the set-valued parts are implicitly discretized.

The rest of the section is dedicated to the well-posedness and stability analysis of the closed-loop system (14)-(19).

It is worth noting that the use of the backstepping-like approach allows us to overcome the maximal monotonicity assumption by splitting the selection process in two steps. Namely, we first compute the selection of the values of ν_k by using (15)-(16). After that, we use (18)-(19) to compute the selection of η_k, considering ν_k fixed. The following lemma formalizes this.

Lemma 10. The closed-loop system (14)-(19) is well-posed in the sense that, for any time t_k, $k \in \mathbb{N}$, there exists a unique selection of the control value. Such value depends on the current state $x(t_k)$ only. Moreover, we have

$$\tilde{x}_{1,k+1} = J_{\mathbf{S}}^{h\gamma_1}(x_{1,k}) \tag{20a}$$
$$\tilde{x}_{2,k+1} - \nu_k = J_{\mathbf{S}}^{h\gamma_3}((1 - h\gamma_2)(x_{2,k} - \nu_k)) \tag{20b}$$

which in fact implies

$$\nu_k = -\gamma_1 \mathcal{S}^{h\gamma_1}(x_{1,k}) \tag{21a}$$
$$\eta_k = -\gamma_3 \mathcal{S}^{h\gamma_3}((1 - h\gamma_2)(x_{2,k} - \nu_k)). \tag{21b}$$

Proof. Consider the subsystem described by (15)-(16). It can be rewritten as $x_{1,k} \in (I + h\gamma_1 \operatorname{Sgn})(\tilde{x}_{1,k+1})$. Recalling the definition of the resolvent of a maximal monotone operator given by (7), it becomes evident that $\tilde{x}_{1,k+1} = J_{\mathbf{S}}^{h\gamma_1}(x_{1,k})$. It follows from (15) that $J_{\mathbf{S}}^{h\gamma_1}(x_{1,k}) = x_{1,k} + h\nu_k$, from where we easily infer (21a) with the aid of (8). Equations (20b) and (21b) follow *mutatis mutandis* by starting the argument with subsystem (18)-(19).

The following technical result will be useful in the upcoming discussion.

Proposition 11. Let $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued maximal monotone operator. Assume that there exists $\rho > 0$ such that $\rho\mathcal{B}_n \subseteq \mathbf{F}(0)$, where $\mathcal{B}_n := \{\xi \in \mathbb{R}^n | \|\xi\| \leq 1\}$. Let $x \in \mathbb{R}^n$, then, $x \in \varepsilon\mathbf{F}(0)$ if and only if, $J_{\mathbf{F}}^{\varepsilon}(x) = 0$. Moreover, if x is such that $x \notin \varepsilon\mathbf{F}(0)$, then

$$\|J_{\mathbf{F}}^{\varepsilon}(x)\| \leq \|x\| - \varepsilon\rho.$$

Proof. The first statement follows directly from (7). On the other hand, assume $x \notin \varepsilon\mathbf{F}(0)$ which in turn implies $\varepsilon\rho x/\|x\| \in \varepsilon\mathbf{F}(0)$. Now, using the non-expansiveness of the resolvent and the previous property, we arrive at

$$\|J_{\mathbf{F}}^{\varepsilon}(x)\| = \left\| J_{\mathbf{F}}^{\varepsilon}(x) - J_{\mathbf{F}}^{\varepsilon}\left(\varepsilon\rho\frac{x}{\|x\|}\right)\right\| \leq \|x\| - \varepsilon\rho.$$

Once the well-posedness of the closed-loop has been established, we turn to the study of the existence of an invariant region.

Lemma 12. Consider the closed-loop system (14)-(19). The set

$$\mathcal{R} := \left\{ x \in \mathbb{R}^2 \mid |x_{1,k}| \leq h\gamma_1, |x_{2,k} - \nu_k| \leq \frac{h\gamma_3}{1 - h\gamma_2} \right\} = \left\{ x \in \mathbb{R}^2 \mid \tilde{x}_{1,k+1} = 0, \tilde{x}_{2,k+1} - \nu_k = 0 \right\} \tag{22}$$

is robustly positively invariant, i.e., it is invariant in the presence of the disturbances $w_{i,k}$, whenever there exist gains $\gamma_i > 0$, $i = 1, 2, 3$, and a fixed sampling time $h > 0$ such that

$$\gamma_1 \geq W_1 + hW_2 \tag{23a}$$
$$\frac{\beta - 1}{\beta h} \geq \gamma_2 \geq \frac{1 - \alpha h}{h} \tag{23b}$$
$$\gamma_3 \geq \alpha(\gamma_1 + W_1 + 2hW_2) \tag{23c}$$

for some $0 < \alpha < \frac{1}{h}$ and $\beta > \frac{1}{\alpha h} > 1$ pre-specified and considered fixed.

Proof. First, notice that (22) holds as a consequence of applying Proposition 11 to (20) and setting $\rho = 1$. We proceed with the proof by showing the positive invariance of $\mathcal{R}$. To this end, assume that $k > 0$ is such that $\tilde{x}_{1,k+1} = 0$ and $\tilde{x}_{2,k+1} - \nu_k = 0$. According to (22), we have to prove that, for all $n_0 \geq 1$, $|x_{1,k+n_0}| \leq h\gamma_1$ and $(1 - h\gamma_2)|x_{2,k+n_0} - \nu_{k+n_0}| \leq h\gamma_3$. From (14a), (15) and (19) we get

$$|x_{1,k+1}| = |\tilde{x}_{1,k+1} + h(\tilde{x}_{2,k+1} - \nu_k) + hw_{1,k} + h^2 w_{2,k}| \leq h(W_1 + hW_2) \leq h\gamma_1, \tag{24}$$

where we have made use of the assumptions $\tilde{x}_{1,k+1} = 0$ and $\tilde{x}_{2,k+1} - \nu_k = 0$ together with (23a). On the other hand, from (14b) and (19) it follows that

$$|x_{2,k+1} - \nu_{k+1}| \leq \gamma_1 |\mathcal{S}^{h\gamma_1}(x_{1,k}) - \mathcal{S}^{h\gamma_1}(x_{1,k+1})| + hW_2,$$

where we have made use of (21a) in the last inequality. Using Proposition 8 together with (14a) and (19) we can establish the bound

$$|x_{2,k+1} - \nu_{k+1}| \leq \gamma_1 + W_1 + 2hW_2.$$

Hence, by noticing that (23b) can be rewritten as $1/\beta \leq 1 - h\gamma_2 \leq \alpha h$ and taking into account (23c) we obtain

$$(1 - h\gamma_2)|x_{2,k+1} - \nu_k| \leq \alpha h (\gamma_1 + W_1 + 2hW_2) \leq h\gamma_3. \tag{25}$$

Finally, it follows from (24) and (25) that $\tilde{x}_{1,k+2} = \tilde{x}_{2,k+2} - \nu_{k+1} = 0$, and the results follows by induction.

We call $\mathcal{R}$ the discrete-time sliding region, and when $x \in \mathcal{R}$ we say that the system is in a discrete-time sliding regime (or phase), see Huber et al. [2016a].

Remark 13. Eventhough the dependence between the gain γ_2 and the sampling time h is such that $\gamma_2 \to +\infty$ as $h \to 0$, the control input (17) remains uniformly bounded with respect to the sampling time h, whenever $x \in \mathcal{R}$. Indeed, given a fixed $0 < \alpha < \frac{1}{h}$ and $\beta > \frac{1}{\alpha h} > 1$, and making use of (23b), (21) and Proposition 8, we arrive at the conclusion that the control input satisfies

$$|u_k| \le \gamma_2 |x_{2,k} - \nu_k| + |\eta_k| \le \beta\gamma_3 .$$

The next results shows that the selection proposed above using the backstepping-like approach makes sense.

Theorem 14. Consider the closed-loop dynamics given by (14)-(19). Let all the assumptions of Lemma 12 hold. Then, the region $\mathcal{R}$ in (22) is finite-time stable, that is, for any initial condition $x(0) \in \mathbb{R}^2$, the closed-loop system trajectories converge to $\mathcal{R}$ in a finite number of steps.

Proof. Let us start considering the Lyapunov-function candidate

$$V(k, x_k) = |\tilde{x}_{1,k+1}| + |\tilde{x}_{2,k+1} - \nu_k|.$$

Notice that proving $\tilde{x}_{1,k+1} \to 0$ and $\tilde{x}_{2,k+1} - \nu_k \to 0$ is equivalent to proving $(x_{1,k}, x_{2,k}) \to \mathcal{R}$. Computation of the difference $\Delta V_k(x) := V(k+1, x_{k+1}) - V(k, x_k)$ gives

$$\Delta V_k(x) = |J_{\mathbf{S}}^{h\gamma_1}(x_{1,k+1})| - |\tilde{x}_{1,k+1}| \\ + |J_{\mathbf{S}}^{h\gamma_3}\left((1-h\gamma_2)(x_{2,k+1} - \nu_{k+1})\right)| - |\tilde{x}_{2,k+1} - \nu_k| \quad (26)$$

We consider several cases.

Case 1 $\left(|x_{1,k+1}| \ge h\gamma_1, |x_{2,k+1} - \nu_{k+1}| \ge \frac{h\gamma_3}{1-h\gamma_2}\right)$**.** Direct application of Proposition 11 to (20) yields

$$|J_{\mathbf{S}}^{h\gamma_1}(x_{1,k+1})| \le |x_{1,k+1}| - h\gamma_1 \quad (27a)$$

$$|J_{\mathbf{S}}^{h\gamma_3}\left((1-h\gamma_2)(x_{2,k+1} - \nu_{k+1})\right)| \le \\ (1-h\gamma_2)|x_{2,k+1} - \nu_{k+1}| - h\gamma_3 . \quad (27b)$$

By substituting (27) into (26) and after some straightforward manipulations involving (14), (15) and (19) one obtains

$$\Delta V_k(x) \le -h\left(\gamma_1 - W_1 - hW_2\right) - h\left(\gamma_2 - 1\right)|\tilde{x}_{2,k+1} - \nu_k| \\ + (1-h\gamma_2)(hW_2 + |\nu_k - \nu_{k+1}|) - h\gamma_3 . \quad (28)$$

Recall that ν_k is given by (21a) and that $\mathcal{S}^\varepsilon$ is Lipschitz continuous (Proposition 8). Thus, (28) transforms into

$$\Delta V_k(x) \le -h(\gamma_2 - 1 - \alpha)|\tilde{x}_{2,k+1} - \nu_k| - h(\gamma_1 - W_1 - hW_2) \\ - h\left(\gamma_3 - \alpha(\gamma_1 + W_1 + hW_2)\right) < 0. \quad (29)$$

Case 2 $\left(|x_{1,k+1}| \ge h\gamma_1, |x_{2,k+1} - \nu_{k+1}| < \frac{h\gamma_3}{1-h\gamma_2}\right)$**.** Equation (27a) still holds and, because of Proposition 11, we have $J_{\mathbf{S}}^{h\gamma_3}\left((1-h\gamma_2)(x_{2,k+1} - \nu_{k+1})\right) = 0$. Hence, equation (26) results in

$$\Delta V_k(x) \le |J_{\mathbf{S}}^{h\gamma_1}(x_{1,k+1})| - |\tilde{x}_{1,k+1}| - |\tilde{x}_{2,k+1} - \nu_k| \\ \le -h\left(\gamma_1 - W_1 - hW_2\right) \\ - (1-h)|\tilde{x}_{2,k+1} - \nu_k| < 0. \quad (30)$$

Case 3 $\left(|x_{1,k+1}| < h\gamma_1, |x_{2,k+1} - \nu_{k+1}| \ge \frac{h\gamma_3}{1-h\gamma_2}\right)$**.** Equation (27b) holds and $J_{\mathbf{S}}^{h\gamma_1}(x_{1,k+1}) = 0$. Hence, equation (26) satisfies

$$\Delta V_k(x) \le -|\tilde{x}_{1,k+1}| + (1-h\gamma_2)|x_{2,k+1} - \nu_{k+1}| - h\gamma_3 \\ - |\tilde{x}_{2,k+1} - \nu_k| \\ \le -|\tilde{x}_{1,k+1}| - h(\gamma_2 - \alpha)|\tilde{x}_{2,k+1} - \nu_k| \\ - h\left(\gamma_3 - \alpha(\gamma_1 + W_1 + 2hW_2)\right) < 0. \quad (31)$$

Case 4 $\left(|x_{1,k+1}| < h\gamma_1, |x_{2,k+1} - \nu_{k+1}| < \frac{h\gamma_3}{1-h\gamma_2}\right)$**.** Both resolvents are zero, that is, $J_{\mathbf{S}}^{h\gamma_1}(x_{1,k+1}) = 0$ and $J_{\mathbf{S}}^{h\gamma_3}\left((1-h\gamma_2)(x_{2,k+1} - \nu_{k+1})\right) = 0$. Hence,

$$\Delta V_k(x) = -|\tilde{x}_{1,k+1}| - |\tilde{x}_{2,k+1} - \nu_k| \le 0. \quad (32)$$

It is now clear that, for any $\tilde{x}_k \notin \mathcal{R}$, the Lyapunov function is strictly decreasing, allowing us to reach the conclusion on asymptotic stability of $\mathcal{R}$. Moreover, the set $\mathcal{R}$ is attained in finite-time. Namely, since ΔV_k is strictly negative outside $\mathcal{R}$, there exists a finite $k^* > 0$ such that Case 4 holds and $\tilde{x}_{1,k^*+2} = 0$ and $\tilde{x}_{2,k^*+2} - \nu_{k^*+1} = 0$ because of Proposition 11.

5. NUMERICAL EXAMPLE

In this section we present the performance of the controller with a numerical example. The performance is assessed by simulations under the assumption that the discrete-time dynamics (14) is a suitable approximation of the continuous-time plant (4) for values of h sufficiently small. The simulations of the closed-loop system (4), (17) were carried out using a zero-order hold as interface between the discrete-time controller and the continuous-time plant.

In all the forthcoming simulations, we set the disturbances as $w_1(t) = 2\sin(3t)\cos(\sqrt{2}t)\sin(\sqrt{5}t - \pi/3)$ and $w_2 = \sin(\sqrt{2}t)$. Hence, $W_1 = 2$ and $W_2 = 1$. Let $h = 5$ ms, and $\alpha = 10$. Satisfaction of (23) requires

$$\gamma_1 \ge 2.005,\ 200 > \gamma_2 \ge 150,\ \gamma_3 \ge 10\gamma_1 + 20.1 . \quad (33)$$

Note that, with this setup we need $\beta \ge 20$, with the actual value depending on γ_2. The initial condition was set to $x_0 = [15, -15]^\top$ and the gains as $\gamma_1 = 3$, $\gamma_2 = 150$ and $\gamma_3 = 50.1$. Figure 1 shows the time evolution of the system state together with the control input and the virtual nominal state $\tilde{x}_k$. The peak in the control happens one instant before the arrival of x to $\mathcal{R}$, which occurs at 4.7 s approximately. The plots are produced by linearly interpolating the sampled signals, this is the reason why the controls do not show the expected stepwise nature.

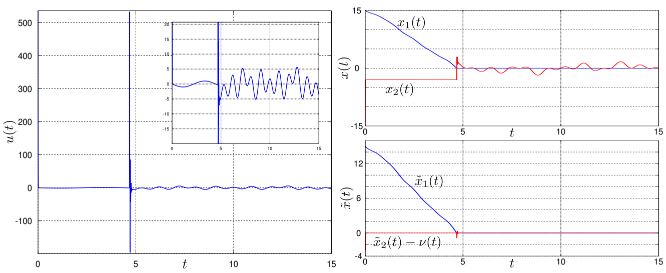

Fig. 1. Time evolution of the system trajectories and the control selection of (4), (17) using the implicit discretization and the backstepping-like algorithm described in Section 4. Gains were set as $\gamma_1 = 3$, $\gamma_2 = 150$ and $\gamma_3 = 50.1$.

The same plant, without any change in the parameters, was simulated using the explicit discretization

$$\hat{u}_k = -\gamma_2\left(x_{2,k} + \gamma_1 \operatorname{sgn}(x_{1,k})\right) \\ - \gamma_3 \operatorname{sgn}\left(x_{2,k} + \gamma_1 \operatorname{sgn}(x_{1,k})\right) . \quad (34)$$

The simulation results are shown in Figure 2. Equation (34) lacks a selection strategy based on the information available at time t_k, which explains the noticeable increase in chattering. The reader familiar with sliding mode control theory could argue that the control law (34) can yield roughly the 'same level' of chattering as the implicit scheme by properly regularizing the signum

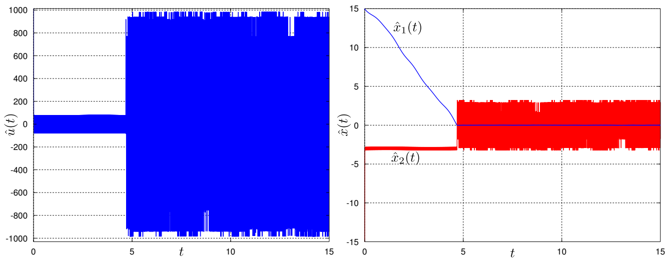

Fig. 2. Time evolution of the system trajectories and control input of the closed loop (4), (34) using the explicit discretization with gains $\gamma_1 = 3$, $\gamma_2 = 150$ and $\gamma_3 = 50.1$.

function. This argument is only partially true, because a regularized control does not guarantee the reduction of chattering [Young et al. 1999]. Namely, the smoothness of the control law will depend on the required precision and the sampling period and, to the best of the authors' knowledge, there is no systematic procedure for choosing the appropriate regularization based on these parameters, just trial and error by simulation, (see Huber et al. [2016b] for an example).

6. CONCLUSIONS AND FURTHER WORK

A set-valued nested controller was proposed. The controller ensures the robust regulation of the output in the presence of non-vanishing mismatched disturbances. The implemented controller uses an implicit discretization together with a backstepping-like algorithm for the selection of the control values. The proposed selection strategy exhibits a better performance when compared with the explicit discretization.

A possible future direction is the study of integral nested controllers which would eliminate the reaching phase, i.e., the period of time taking place before the state enters $\mathcal{R}$. This would avoid the large initial peaks displayed by the control input. Also, the methodology can be extended to more general classes of systems with order larger than 2.

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