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PROJECTED DYNAMICS OF CONSTRAINED HAMILTONIAN SYSTEMS*

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***Abstract*— A novel formulation for the description of implicit port-Hamiltonian control systems is proposed and its potential use for the design of the control laws stabilizing a given submanifold described as a zero level set of an admissible energy function is shown. Using the developed formulation, a number of results on the stabilization of port-Hamiltonian systems are presented. The obtained results are formulated in a way that allows for direct application.**

I. INTRODUCTION

The Hamiltonian formalism is a common tool to describe the dynamics of a wide class of mechanical, electrical and thermodynamic systems, see, e.g., [1], [2], [3], [4], [5] ones.

In many cases there are constraints imposed on the system coordinates. These constraints reflect the internal structure of the system, for instance, rigid connections between the system's elements. There are two types of constraints: holonomic and non-holonomic ones, [1], [6]. The action of the former results in restricting the system's evolution to a submanifold of the state space while the latter restrict the system's dynamics without confining it to a subset of the state space.

When the system is subject to the action of external forces it is common to consider a pair of (energy-adjoint) port variables (u, y) such that their product is equal to the power supplied into the system. Such model is referred to as a port-Hamiltonian system (see [7] for the original definition).

In the following, we consider the holonomic case that corresponds to restricting the system's evolution to a manifold in the state space. In general, there are two different approaches: the explicit representation such that the dynamics has the form of an ordinary differential equation on the manifold and the implicit representation with the dynamics described by a set of differential-algebraic equations evolving in a Euclidean space (see, e.g., [8] for a related discussion on constrained Hamiltonian systems). There has been a lot of research on the analysis and control of explicit systems [9], [10]. However, there are few results on the control of Hamiltonian systems in implicit formulation.

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Despite the lack of attention, the implicit formulation possesses certain useful features. For instance, when writing a holonomic Hamiltonian system in the implicit form, in most cases the Hamiltonian function turns out to be separable, i.e., it can be represented as a sum of kinetic energy with a constant inertia matrix and potential energy (cf. Assumption **A3.** in Sec. II-B). In a recent paper, [11], an implicit port-Hamiltonian representation of mechanical systems was considered from a control perspective. In particular, an approach to energy shaping as well as a result on stabilization of homoclinic orbits were presented. The same paper contains an extensive discussion about the advantages of using an implicit model.

In this paper we aim at developing a uniform approach to the modeling and stabilization of implicit port-Hamiltonian systems thus expanding results presented in [11]. The obtained results are formulated in a way that allows for direct application. All necessary operations are reduced to performing algebraic operations that can be easily numerically implemented. On the other hand, this approach reveals some inherent geometric structure of the systems under study.

The proposed framework can be extended in a number of ways, in particular, to include the case of virtual holonomic constraints (VHC) that have become a common tool in designing control laws for mechanical systems, see, e.g., [12], [13].

The paper is organized as follows. In Section II, the definition of an implicit port-Hamiltonian system along with a number of regularity assumptions is given. Also, a number of preliminary results are presented. Section III presents a novel formulation of the system's dynamics based on the use of a particular projection operator, whose properties are discussed in detail. In Section IV, the problem of stabilization of the level set of a generalized energy function is addressed and a number of examples are provided. The paper is concluded with a brief summary and an outline of the future work.

II. PROBLEM STATEMENT

A. Notation

We use the following notation: a square matrix $A \in \mathbb{R}^{n\times n}$ is interpreted both as a real matrix and as a coordinate representation of a linear operator acting in the standard Euclidean basis; $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the *range* and the *null space* of the linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$. Likewise, if $\{v_1, \dots, v_q\}$ is a set of vectors $v_i \in \mathbb{R}^n$, not necessarily linearly independent, and $V = [v_1, \dots, v_q]$ is a matrix formed by these vectors, $\mathcal{R}(V)$ and $\mathcal{N}(V)$ denote the column space of V and the set of vectors orthogonal to v_i, $i = 1, \ldots, q$. Furthermore, $\operatorname{co}\mathcal{R}(V)$ and $\operatorname{co}\mathcal{N}(V)$ denote the orthogonal complements to $\mathcal{R}(V)$ and $\mathcal{N}(V)$, respectively.

The symbol $\oplus$ denotes the direct sum of two subspaces. Let V and W be two subspaces of $\mathbb{R}^n$, $V \cap W = \{0\}$, every $z \in V \oplus W$ can be uniquely represented as a sum $z = z_V + z_W$ such that $z_V \in V$ and $z_W \in W$.

B. An implicit Hamiltonian model

We consider a controlled mechanical system with Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$. Let there be $k < n$ *holonomic constraints* $c(r) = 0$, $c : \mathbb{R}^n \to \mathbb{R}^k$, restricting the configuration space of the system to an $(n-k)$-dimensional submanifold Γ of the configuration space $\mathbb{R}^n$. Using the Hamiltonian formalism, the dynamics of this system are described by:

$$\begin{bmatrix} \dot r \\ \dot p \end{bmatrix} = J\left(\nabla H(x) + \nabla c(r)\mu\right) + F(x)u \tag{1a}$$

$$y = \nabla^\top H(x)\, F(x) \tag{1b}$$

$$0 = c(r)\ , \tag{1c}$$

where $x^\top = \begin{bmatrix} r^\top & p^\top \end{bmatrix} \in \mathbb{R}^{2n}$ is the state, r and p are the positions and the momenta,

$$\nabla c(x) = \begin{bmatrix} \nabla_r c(r) \\ 0 \end{bmatrix}$$

is the gradient of $c(r)$, $\mu \in \mathbb{R}^{*k}$ is the vector of *implicit* variables that enforce the holonomic constraints, $(u, y) \in \mathbb{R}^{*m} \times \mathbb{R}^m$ are the conjugated external port variables, and $F(x) = \begin{bmatrix} 0 \\ f(r) \end{bmatrix}$, where $f(r)$ is a $(n \times m)$-matrix, is the control vector field. Note that we write $F(x)$ and $\nabla c(x)$ when referring to the matrices with $2n$-dimensional columns. We also assume that all functions are sufficiently smooth and the gradient is a column vector. In particular, if $a(x)$ is a vector-valued function, $\nabla_x a(x)$ is defined to be the transposed Jacobi matrix.

Equation (1c) constrains the configuration space of (1). From the geometrical viewpoint, the constraint (1c) defines a smooth (configuration) manifold $Q = \{x \in \mathbb{R}^{2n} | c(x) = 0\}$, which turns out to be an embedded manifold of co-dimension k as will be formally defined later on. We wish to ensure that the mentioned constraints are preserved by the system dynamics. To do so we require $c(r)$ to be invariant w.r.t. (1a), i.e., $\dot c = 0$. This yields the so-called *hidden* (*secondary*) constraints,

$$G(x) = \nabla^\top c(x) J \nabla H(x) = 0\ , \tag{2}$$

which have to hold for $x \in Q$. That is to say, the hidden constraints are to be *Q-regular*.

Note that the hidden constraints depend neither on μ nor on u as follows from the fact that the holonomic constraints are formulated for the positions r. The following result is directly related to the notion of the hidden constraint (2).

Proposition 1: For system (1), $\nabla H(x)$ is in the null space of $J\nabla c$, i.e., $\nabla H \in \mathcal{N}(J\nabla c)$ for all $x \in \mathcal{M}$.

Proof: Write (2) as a scalar product,

$$\langle \nabla c, J\nabla H\rangle = -\langle J\nabla c, \nabla H\rangle = 0,$$

whence the result follows. Note that the preceding identity may not hold outside of $\mathcal{M}$ as the equality (2) may not be satisfied. ∎

The above implies that the Hamiltonian is invariant under the action of the vector field of constraint forces. Furthermore, we have $\langle \nabla c(x), J\nabla c(x)\rangle = 0$ and $\langle \nabla c(x), F(x)\rangle = 0$, which implies that both the internal forces vector field and the control vector field are tangential to the submanifold Γ.

Now, considering $T^*\mathbb{R}^n$ as the state space manifold, we observe that (1) evolves on a submanifold $\mathcal{M} \subset T^*\mathbb{R}^n$,

$$\mathcal{M} = \{x = (r, p) \in \mathbb{R}^{2n} | c(x) = 0, G(x) = 0\}\ .$$

All results formulated below will hold for $x \in \mathcal{M}$. In particular, we require the following assumptions to hold for every $x \in \mathcal{M}$:

A1. The constraints are Q-regular, i.e., $\operatorname{rank}(\nabla c(r)) = k$ for all r such that $c(r) = 0$.

A2. The initial conditions belong to $\mathcal{M}$, i.e., $x(0) = (r(0), p(0)) \in \mathcal{M}$.

A3. The energy is *separable* and *positive definite w.r.t.* p, i.e.,

$$H(x) = P(r) + K(p),$$
$$K(p) = \frac{1}{2} p^\top M^{-1} p,\ M = M^\top > 0\ ,$$

where P and K are the potential and kinetic energy, respectively.

Assumptions **A1** and **A3** guarantee that $\mathcal{M}$ is a proper subbundle of $T^*\mathbb{R}^n$. In particular, we observe that ∇c and ∇G are linearly independent and the tangent space to $\mathcal{M}$ is given by $T_x\mathcal{M} = \mathcal{N}(\operatorname{span}(\nabla c(x), \nabla G(x)))$.

Remark 1: Note that the inertia matrix M in Assumption **A3** does not depend on r thus rendering kinetic energy independent of the coordinates r. This is one of the main advantages of the proposed framework (see [11] for a detailed discussion).

Remark 2: Note that $\nabla c(x)$ and $\nabla G(x)$ live in the cotangent space to $T^*\mathbb{R}^n$ at x, i.e., $\nabla c(x) \in T^*_x(T^*\mathbb{R}^n)$ and $\nabla G(x) \in T^*_x(T^*\mathbb{R}^n)$; the components of $\nabla c(x)$ and $\nabla G(x)$ thus define differential 1-forms whose kernel corresponds to the set of admissible vector fields when moving along Γ.

To be more precise, we add that $\nabla c(r)$ is actually defined on the cotangent bundle to the configuration manifold Q, hence $\nabla c(r) \in T^*Q$. However, it can be naturally considered to belong to the cotangent bundle of $\mathcal{M}$ (see the Eq. just below (1)). In the following, we will assume that the interpretation of ∇c is clear from the context.

Proposition 2: Let Assumption **A1** and **A3** hold. Then there exist implicit variables μ such that the constraints (1c) and (2) are satisfied. Furthermore, the variables μ are uniquely defined.

Proof: To see this we compute the derivative of the vector of hidden constraints G w.r.t. (1a) to get

$$\dot G = \nabla^\top G \cdot J\nabla H + \nabla^\top G \cdot J\nabla c\mu + \nabla^\top G \cdot F(x)u = 0\ , \tag{3}$$

which implicitly defines μ as a function of x and u. Notice that the $(k \times k)$-matrix

$$\nabla^\top G \cdot J\nabla c = -\nabla_r^\top c(r)\, M^{-1}\, \nabla_r c(r) \tag{4}$$

is negative definite as follows from Assumptions **A1** and **A3** and hence, invertible. This ensures the well-posedness of the problem. ∎

Remark 3: Note that while the vectors $\{\nabla c^i\}$ and $\{\nabla G^i\}$ are linearly independent as was discussed above, the vectors $\{J\nabla c^i\}$ and $\{\nabla G^i\}$ can be linearly dependent.

Finally, Assumption **A2** implies that the initial conditions are consistent with the constraints.

III. PROJECTED SYSTEM DYNAMICS

Theorem 3: Let Assumptions **A1 – A3** hold. Then the system (1) can be represented as

$$\begin{aligned} \dot{x} &= PJ\nabla H + PF(x)u, \\ y &= \nabla^\top H(x)\, PF(x) \end{aligned} \tag{5}$$

where

$$P = I - J\nabla c \left(\nabla^\top G \cdot J\nabla c\right)^{-1} \nabla^\top G \tag{6}$$

is a $[2n \times 2n]$ projection operator.

Proof: We have the system (1a) and the constraint (3), where the $(k \times k)$-matrix $\nabla^\top G \cdot J\nabla c$ is full rank. Hence, we can express μ from (3) to get

$$\mu = -\left(\nabla^\top G \cdot J\nabla c\right)^{-1} \left(\nabla^\top G \cdot J\nabla H + \nabla^\top G \cdot F(x)u\right)$$

and substitute it into (1a):

$$\begin{aligned} &\dot{x} = J\nabla H(x) + F(x)u - \\ &J\nabla c(r) \left(\nabla^\top G {\cdot} J\nabla c\right)^{-1} \left[\nabla^\top G {\cdot} J\nabla H + \nabla^\top G \cdot F(x)u\right] \\ &\quad = \left(I - J\nabla c(r) \left(\nabla^\top G \cdot J\nabla c\right)^{-1} \nabla^\top G\right) J\nabla H + \\ &\qquad \left(I - J\nabla c(r) \left(\nabla^\top G \cdot J\nabla c\right)^{-1} \nabla^\top G\right) F(x)u. \end{aligned}$$

Defining $P = I - J\nabla c(r) \left(\nabla^\top G J\nabla c\right)^{-1} \nabla^\top G$ we arrive at the required result. Note that P is a projection operator as follows from $P^2 = P$. ∎

This result implies that the introduction of the constraint is equivalent to projecting the right-hand sides onto some linear subspace of the tangent space using the projection operator P. One may notice that the projection P is oblique (non-orthogonal) since $P \neq P^\top$. Thus, $\mathcal{R}(P) \neq \mathcal{N}(P)^\perp$. The following theorem provides a detailed characterization of P.

Theorem 4: When acting from the left, the null and the range space of the projection operator P are given by $\mathcal{R}(J\nabla c)$ and $\mathcal{N}(\nabla^\top G)$, respectively. When P is acting from the right, the range and the null spaces are $\mathcal{N}\left((J\nabla c)^\top\right)$ and $\mathcal{R}(\nabla G)$.

Proof: Consider P when acting from the left. We have $PJ\nabla c = 0$, whence $v \in \mathcal{R}(J\nabla c) \Rightarrow v \in \mathcal{N}(P)$. This implies $\mathcal{R}(J\nabla c) \subset \mathcal{N}(P)$. Let, furthermore, $v \in \mathcal{N}(\nabla^\top G)$, i.e., $\nabla^\top G \cdot v = 0$. We have $Pv = v$, i.e., $P\big|_{\mathcal{N}(\nabla^\top G)} = \mathrm{id}$, whence $v \in \mathcal{N}(\nabla^\top G) \Rightarrow v \in \mathcal{R}(P)$ and hence, $\mathcal{N}(\nabla^\top G) \subset \mathcal{R}(P)$.

Finally, let $v \in \mathcal{R}(P) \cap \mathcal{N}(P)$. Thus, $v = Pw$ and $Pv = 0$. Using the idempotence property of the projection operator, $P^2 = P$, we get $0 = Pv = Pw = v$, i.e., $v = 0$. This implies that $\mathcal{R}(P) \oplus \mathcal{N}(P) = \mathbb{R}^{2n}$. Noting that $\dim(\mathcal{R}(J\nabla c)) = k$ and $\dim(\mathcal{N}(\nabla^\top G)) = 2n - k$, as follows from Assumptions **A1** and **A3**, we conclude that $\mathcal{R}(J\nabla c) \oplus \mathcal{N}(\nabla^\top G) = \mathbb{R}^{2n}$, whence we get the desired characterization.

The case when P acts from the right can be shown similarly by considering the transpose of the projection operator, $P^\top = I - \nabla^\top G \left((J\nabla c)^\top \nabla G\right)^{-1} (J\nabla c)^\top$, and performing the same analysis as above. ∎

Corollary 5: The result of Theorem 4 can be summarized using the following diagram:

$$\begin{array}{ccccc} \mathcal{N}(J\nabla c) & & \mathcal{N}(\nabla^\top G) & & Range \\ \oplus & : P : & \oplus & & --- \\ \mathcal{R}(\nabla^\top G) & & \mathcal{R}(J\nabla c) & & Null \\ ---- & & ---- & & \\ \mathbb{R}^{2n*} & & R^{2n} & & \end{array}$$

E.g., when P acts from the left, every $v \in \mathbb{R}^{2n}$ can be uniquely written as

$$v = v_\parallel + v_\perp$$

s.t. $v_\parallel = Pv \in \mathcal{N}(\nabla^\top G)$, and $v_\perp = v - Pv \in \mathrm{span}(J\nabla c_i)$.

Actually, P is defined on $T(T^*\mathbb{R}^n)$ when acting from the left and on $T^*(T^*\mathbb{R}^n)$ when acting from the right. The product $xPy = \langle x, Py\rangle = \langle P^\top x, y\rangle$ thus corresponds to the standard vector-covector contraction $T^*(T^*\mathbb{R}^n) \times T(T^*\mathbb{R}^n) \mapsto \mathbb{R}$.

We show that the formulation (5)-(6) agrees with the standard results described in Sec. II. First, consider

$$\dot{c} = \nabla c^\top(x) PJ\nabla H + \nabla c^\top(x) PF(x)u.$$

Note that $\nabla c^\top(x) \in \mathcal{N}(J\nabla c)$, which implies $\nabla c^\top(x) P = \nabla c^\top(x)$. Finally, we observe that $\langle \nabla c(x), F(x)\rangle = 0$, whence $\dot{c} = \nabla c^\top(x) J\nabla H = G(x)$. Next, we differentiate $G(x)$ to get

$$\dot{G} = \nabla^\top G(x) PJ\nabla H + \nabla^\top G(x) PF(x)u.$$

Theorem 4 implies that $\nabla^\top G(x) P = 0$ hence, $\dot{G} = 0$ as expected.

Following the same line we can show the following result.

Lemma 6: The passive output $y = \nabla^\top H(x)\, PF(x)$ coincides with the original one defined in (1) along $\mathcal{M}$.

Proof: According to Prop. 1, $\nabla^\top H(x) \in \mathcal{N}(J\nabla c)$ which implies that $\nabla^\top H(x)\, P = \nabla^\top H(x)$, whence the result follows. ∎

IV. STABILIZATION OF THE LEVEL SET OF A GENERALIZED ENERGY FUNCTION

The classical approach to the control of mechanical systems is based on energy considerations, [14], [15], [16], [17]. A typical version of this approach consists in using the control to stabilize the level set of the energy function corresponding to a desired operation mode, $H(x) = \bar{H}$. For instance, a mechanical system can be driven towards an open-loop unstable equilibrium by forcing its trajectories to converge to an energy level-set that includes such equilibrium. In practice, this amounts to choosing the control in order to ensure negative-definiteness of a control Lyapunov function (CLF) of the form $V(x) = (H(x) - \bar{H})^2$, which is positive everywhere except the respective level of $H(x)$.

Once the system state is sufficiently close to the unstable equilibrium, the controller can be switched to a linear law that stabilizes it (see [18], [19], [20]). However, for complex mechanical systems the level set of the Hamiltonian function can contain orbits that do not pass through the desired unstable equilibrium, thus rendering the proposed control strategy ineffective.

One possible solution to overcome such problem is to further restrict the set to be stabilized by restricting the system dynamics to the intersection of the level sets of some extra functions $\vartheta_i(x)$, $i = 1, \ldots, q$, i.e., requiring $\vartheta_i(x) = \bar{\vartheta}_i$. Note that, for some i, $\vartheta_i(x)$ may coincide with the Hamiltonian $H(x)$. The respective CLF would then take the form $V(x) = \sum_{i=1}^{q}(\vartheta_i(x) - \bar{\vartheta}_i)^2$.

However, it may happen that some of the introduced functions are not invariant under the action of system dynamics. Thus the described task falls into two steps: first, determine the stabilizing control law that ensures $\dot{V} < 0$, then switch to the control law that turns the function $V(x)$ into the first integral.

Before proceeding to the formal results we introduce some notations that will be useful in the sequel. Let $\mathcal{L}_0 \subset \mathcal{M}$ be a smooth embedded manifold. Let $V : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function such that $V(x)$ is equal to zero for all $x \in \mathcal{L}_0$ and is positive otherwise. We define the set

$$\mathcal{L}_V = \{x \in \mathcal{M} \setminus \mathcal{L}_0 \mid \nabla^\top V(x) P F(x) = 0\}.$$

Theorem 7: Let $\mathcal{L}_0 \subset \mathcal{M}$ and $V : \mathbb{R}^{2n} \to \mathbb{R}$ be defined as above. Let, furthermore, $V(x) \to \infty$ as $\mathrm{dist}(x, \mathcal{L}_0) \to \infty$. Then the zero level set of $V(x)$, $\mathcal{L}_0$, can be rendered attractive by an appropriate choice of the control if the following holds:

1) The set $\mathcal{L}_V$ does not contain whole trajectories of the uncontrolled system $\dot{x} = PJ\nabla H(x)$.
2) For every $x \in \mathcal{L}_0$, either $V(x)PF(x) \neq 0$ or $V(x)PJ\nabla H(x) = 0$.

We will call that the function V satisfying the requirements of Theorem 7 an *admissible energy-like function* or a *generalized energy function* for the system (5).

The respective differential equation for V is

$$\dot{V} = \nabla^\top V P J \nabla H + \nabla^\top V P F(x) u. \tag{7}$$

For a given admissible energy-like function V, the control law can be determined using the speed gradient method as described in [14], [21]. Let Ψ be a positive definite symmetric matrix, the control can be determined from the differential equation

$$\frac{du}{dt} = -\Psi F^\top(x) P^\top \nabla V.$$

Note that $F^\top(x)P^\top\nabla V$ is the gradient of the full derivative of V taken with respect to u.

The preceding theorem does not tell the way to choose an admissible energy-like function. However, for a given function $V(x)$ one can determine whether the conditions of Theorem 7 hold by carrying out certain geometric analysis involving checking whether $\nabla^\top V$ belongs to some linear subspace defined by ∇G and $J\nabla c$ or not. This will be illustrated in Examples 1 and 2 below. But first we make some observations that will be useful in the sequel.

Lemma 8: The equality $\nabla^\top V P F(x) \neq 0$ holds if the following two conditions take place:

1) $F(x) \notin \mathcal{R}(J\nabla c(r))$,
2) $\nabla^\top V P \notin \mathcal{N}(F(x))$.

Remark 4: The condition $F(x) \notin \mathcal{R}(J\nabla c(r))$ is equivalent to checking linear independence of $f^i(r)$ and $\nabla_r c(r)$ for at least one $i = 1, \ldots, m$. The second condition, $\nabla^\top V P \notin \mathcal{N}(F(x))$ is more difficult to formalize. However, it clearly excludes the case when V depends only on r. This situation may occur when the control goal consists in imposing on the system virtual holonomic constraints.

Example 1: Let $V(x) = H(x)$. According to Prop. 1, $\nabla H \in \mathcal{N}(J\nabla c)$ which implies that $\nabla H \in \mathcal{R}(P)$ according to Theorem 4. For a projection operator we have $v \in \mathcal{R}(P) \Rightarrow Pv = v$, as follows from the idempotency of P. Thus, $\nabla^\top H \cdot P = \nabla^\top H$, which implies that the first term in (7) turns into $\nabla^\top H\, J \nabla H$ which is equal to zero as J is skew symmetric.

Example 2: Consider the generalized energy function from [11]:

$$V(x) = \frac{k_H}{2}\left(H(x) - H_0\right)^2 + k_z f(z) + \frac{k_y}{2}\|y\|^2, \tag{8}$$

where $z(x) : \mathbb{R}^n \times \mathbb{R}^{*n} \to \mathbb{R}^m$ is such that $\nabla z(x) = JF(x)$ and $f(z)$ is a positive definite, radially unbounded function of $z(x)$. The function $V(x)$ is equal to zero on the subset

$$\mathcal{L}_0 = \{x \in \mathcal{M} | H(x) = H_0 \vee z(x) = 0 \vee y(x) = 0\}. \tag{9}$$

We wish to analyze whether the requirements of Theorem 7 are satisfied. To do so, we write (8) as a sum:

$$V(x) = k_H V_H(H) + k_z V_z(z) + k_y V_y(y)$$

and consider the respective components separately. Note that for a composite function $V_w(w(x))$, $V_w : \mathbb{R}^q \to \mathbb{R}$, we have $\nabla_x^\top V_w(w(x)) = \nabla_w^\top V_w \cdot \nabla_x^\top w(x)$, where $\nabla_w^\top V_w$ is $[1 \times q]$ and $\nabla_x^\top w(x)$ is $[q \times 2n]$.

1. $V_H(H(x)) = \frac{1}{2}\left(H(x) - H_0\right)^2$ and $\nabla^\top V_H(H(x)) = \left(H(x) - H_0\right)\nabla^\top H(x)$. This case is discussed in Example 1. Using the results presented in Example 1 we obtain the following characterization:

1) $\nabla^\top V_H \cdot P \cdot F(x) = \left(H(x) - H_0\right)\nabla^\top H \cdot P \cdot F = \left(H(x) - H_0\right) y$. This component is non-zero for all x such that $y(x) \neq 0$ and $H(x) \neq H_0$.
2) $\nabla^\top V_H \cdot P \cdot J\nabla H = \left(H(x) - H_0\right)\nabla^\top H\, P \cdot J\nabla H = 0\ \forall x \in \mathcal{M}$. Hence, $V_H(x)$ is an integral of (1).

2. $V_z(z) = f(z)$. The vector-valued function $z(x)$ satisfies $\nabla z(x) = JF(x)$. This implies, in particular, that

$$\dot{z} = \nabla^\top z \cdot J\nabla H = F^\top(x)\nabla H = y^\top.$$

We have the following:

1) $\nabla^\top V_z \cdot P \cdot F(x) = \nabla_z^\top f(z) F^\top JF(x) = 0 \ \forall x \in \mathcal{M}$. The respective component vanishes identically. This result reflects the fact that in a Hamiltonian system one cannot influence the coordinates directly as their first derivatives do not contain the control u.
2) $\nabla z \in \mathcal{N}(J\nabla c)$ as follows from

$$\nabla^\top z \cdot J\nabla c = F^\top(x) J^\top J \nabla c = F^\top(x)\nabla c = 0.$$

Hence, $\nabla^\top z \cdot P = \nabla^\top z$ and we get

$$\nabla^\top V_z P J\nabla H = \nabla_z^\top f(z)\nabla^\top z(x) J\nabla H = \nabla_z^\top f(z) y^\top,$$

which is equal to 0 if $y = 0$. This implies that the invariance condition holds for $x \in \mathcal{L}_0$.

3. $V(y) = \frac{1}{2}\|y\|^2$ and $\nabla^\top V(y) = y \cdot \nabla^\top y$. Next, we have

$$\nabla y = \begin{pmatrix} \nabla_r \left(p^\top M^{-1} f(r)\right) & M^{-1} f(r) \end{pmatrix}.$$

It holds that $\nabla y \notin \mathcal{R}(\nabla G)$ if

$$\operatorname{span}(\nabla_r c(r)) \cap \operatorname{span}(f(r)) \neq 0. \tag{10}$$

If this condition holds for all $x \in \mathcal{M}$, we have

1) $\nabla^\top V_y \cdot P \cdot F \neq 0$ for $x \in \mathcal{M}$ such that $y(x) \neq 0$ and is equal to 0 otherwise.
2) $\nabla^\top V_y \cdot P \cdot J\nabla H = 0$ for $x \in \mathcal{M}$ such that $y(x) = 0$ hence, for $x \in \mathcal{L}_0$.

We now summarize the preceding analysis.

- The set $\mathcal{L}_0$, (9), is invariant under the action of (1).
- If (10) holds for all $x \in \mathcal{M} \setminus \mathcal{L}_0$, the system is controllable everywhere except the set where $y(x) = 0$. An additional analysis is to be performed in order to determine whether this set contains an invariant subset.

Note that the analysis carried out above turns out to be more general as it imposes less conditions in order to guarantee that the set (9) can be rendered attractive by an appropriate choice of the control u. The reason for this is that the analysis in [11] was confined to a slightly different problem. Namely, the consistency conditions were written as a scalar product $\langle y, f(\hat{u}, \hat{\mu})\rangle$ which had to be equal to 0. The authors then went on by requiring $f(\hat{u}, \hat{\mu}) = 0$, which is overly restrictive: in fact it would suffice to require $f(\hat{u}, \hat{\mu}) \in \mathcal{R}^\perp(y)$. Depending on the dimension of y, its orthogonal complement can be sufficiently large thus leaving more freedom in determining $\hat{u}$. On the other hand, the approach taken in [11] allowed to formulate the stabilizing control as a multiple of the passive output y. In particular, it was shown that the original passive output y can be used as the passive output of the modified system.

V. CONCLUSIONS

We presented a novel formulation for the description of implicit port-Hamiltonian control systems and shown its potential use for designing control laws stabilizing a given submanifold described as a zero level set of an admissible energy function. This result can be extended in a number of directions. In particular, we plan to further exploit the geometric structure of the projection operator P in order to derive more efficient control laws.

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