

# Proportional Plus Integral Control for Set-Point Regulation of a Class of Nonlinear RLC Circuits

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**Abstract** In this paper we identify graph-theoretic conditions which allow us to write a nonlinear RLC circuit as port-Hamiltonian with constant input matrices. We show that under additional monotonicity conditions on the network's components, the circuit enjoys the property of relative passivity, an extended notion of classical passivity. The property of relative passivity is then used to build simple, yet robust and globally stable, proportional plus integral controllers.

**Keywords** Nonlinear networks · Passivity · Port-Hamiltonian systems · Stability · Stabilization

## 1 Introduction

In this paper we look at the problem of global output regulation of nonlinear RLC networks using proportional plus integral (PI) controllers. Besides the simplicity and

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widespread popularity of these controllers, it is well known that PI control is robust *vis-à-vis* parameter uncertainty—due to the integral action that is necessary to reject constant disturbances, even in the nonlinear context [5]. The main contribution of this paper is to show that for a large class of nonlinear RLC circuits with regulated voltage and current sources, we can exploit the property of passivity to ensure that the problem in question can be solved with a simple PI controller around the sources' port variables. This work is a sequel of [14], where we investigated the important property of *relative passivity* of general nonlinear systems and applied it to PI stabilization of a restricted class of RLC circuits.

In this paper we show that a very large class of nonlinear RLC circuits enjoys the relative passivity property and can, therefore, be stabilized via PI control. In particular, the class admits a network fully consisting of nonlinear inductors, nonlinear capacitors and nonlinear resistors. The ferroelectric ceramic capacitor model used in [12] is an example of nonlinear capacitors that are admissible in our circuits. The nonlinear inductor model studied in [11] for main harmonic reduction circuits also satisfies our assumptions. Another admissible and popular nonlinear element is a transformer with nonlinear self-inductance and mutual inductance.

Instrumental for this work is the use of port-Hamiltonian models to describe the RLC networks. This allows us to establish in a straightforward manner the property of relative passivity and identify some additional assumptions on the characteristic functions of the circuit elements that are sufficient to make the stability result global.

The search for energy- (and power-) based Hamiltonian models for physical systems with external ports and (possibly nonlinear) dissipation is interesting in its own right, and it has attracted considerable attention in the last decades. One way to obtain such models is to start from a Brayton–Moser [3, 4] or a Lagrangian formulation and then perform the necessary transformations to arrive at a Hamiltonian description [2, 8]. A critical assumption in this procedure is that the characteristics of the network components are bijective. To avoid this limitation we prefer, in the spirit of network modeling, to proceed from the port-Hamiltonian lossless models of [18], see also [1], and add the required sources and dissipation terminations.

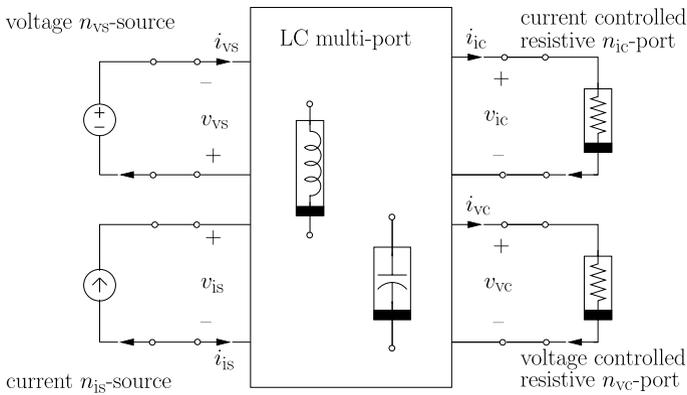
## 2 Port-Hamiltonian Formulation of Nonlinear RLC Networks

The purpose of this section is to set the energy-based models that will be central in the subsequent analysis. A direct constructive method for obtaining Hamiltonian models for LC circuits has been proposed in [1]. Following the suggestion of [18], in this section we extend this method by adding ports to account for voltage and current sources and resistive elements (the inclusion of resistive ports was independently proposed by B. Maschke and published in [17] without a proof). This is shown in Fig. 1.

We consider RLC networks satisfying the following assumptions.

**Assumption 2.1** Capacitors are charge controlled and inductors are flux controlled with characteristics given by

$$v_q = \hat{v}_q(q) \quad \text{and} \quad i_\phi = \hat{i}_\phi(\phi), \quad (2.1)$$



**Fig. 1** An RLC network with sources and dissipation elements viewed as ports

where  $q, v_q \in \mathbb{R}^{n_q}$  are the capacitors’ charges and voltages, and  $\phi, i_\phi \in \mathbb{R}^{n_\phi}$  are the inductors’ fluxes and currents. Furthermore,  $\hat{v}_q(q)$  and  $\hat{i}_\phi(\phi)$  have symmetric Jacobians—that is, they are gradients of scalar functions.

*Remark 2.2* The vector-valued model (2.1) is not restricted to 2-terminal components only; it can describe 4-terminal elements as well, such as nonlinear transformers.

The first condition of the assumption allows us to write the electric and magnetic energies as functions of  $q$  and  $\phi$ , i.e.,

$$\mathcal{E}_q(q) = \int_0^q \hat{v}_q^\top(\tau) d\tau + \mathcal{E}_q(0) \quad \text{and} \quad \mathcal{E}_\phi(\phi) = \int_0^\phi \hat{i}_\phi^\top(\tau) d\tau + \mathcal{E}_\phi(0), \quad (2.2)$$

while the second one guarantees that these functions do not depend on the integration path. This is of course automatically satisfied if the energy-storing elements are all single-port. To define the rest of the network components we denote by

$$v_{ic} = \hat{v}_{ic}(i_{ic}), \quad i_{ic}, v_{ic} \in \mathbb{R}^{n_{ic}} \quad (2.3)$$

and

$$i_{vc} = \hat{i}_{vc}(v_{vc}), \quad v_{vc}, i_{vc} \in \mathbb{R}^{n_{vc}} \quad (2.4)$$

the current and voltage controlled resistors. Time-varying voltage and current sources are represented by  $v_{vs} \in \mathbb{R}^{n_{vs}}$  and  $i_{is} \in \mathbb{R}^{n_{is}}$  respectively.

**Assumption 2.3** The graph  $\mathcal{G}$  associated to the network has a tree  $\mathcal{T}$  containing all capacitors, voltage sources and current controlled resistors. For future reference, denote by  $\mathcal{L}$  the set of links corresponding to  $\mathcal{T}$ .

This assumption excludes loops formed exclusively by capacitors and/or voltage sources as well as cut sets formed exclusively by inductors and/or current sources.

This in turn means that we can choose  $\phi$ ,  $q$ ,  $v_{vs}$  and  $i_{is}$  independently without violating Kirchoff’s laws.

It is possible to write Kirchoff’s voltage law in compact form as  $Bv = 0$ , where  $B$  is the fundamental loop matrix and  $v$  is the vector of branch voltages [9]. Moreover, if  $v$  is partitioned as  $v = \text{col}(v_{\mathcal{L}}, v_{\mathcal{T}})$ , where  $v_{\mathcal{L}}$  and  $v_{\mathcal{T}}$  are the branch voltages of  $\mathcal{L}$  and  $\mathcal{T}$  respectively, then  $B$  takes the form  $B = [I \ F]$  (with  $I$  and  $F$  of appropriate dimensions). If we further partition  $v_{\mathcal{L}}$ ,  $v_{\mathcal{T}}$  and  $F$  as

$$v_{\mathcal{L}} = \begin{bmatrix} v_{\phi} \\ v_{is} \\ v_{vc} \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ v_{is} \\ v_{vc} \end{bmatrix}, \quad v_{\mathcal{T}} = \begin{bmatrix} \hat{v}_q(q) \\ -v_{vs} \\ \hat{v}_{ic}(i_{ic}) \end{bmatrix} \tag{2.5}$$

(the negative sign of the voltage sources indicates that voltage drops across these branches are opposite to current flow) and

$$F = \begin{bmatrix} F_{\phi-q} & F_{\phi-vs} & F_{\phi-ic} \\ F_{is-q} & F_{is-vs} & F_{is-ic} \\ F_{vc-q} & F_{vc-vs} & F_{vc-ic} \end{bmatrix} \tag{2.6}$$

then we can write

$$\dot{\phi} = -F_{\phi-q} \hat{v}_q(q) + F_{\phi-vs} v_{vs} - F_{\phi-ic} \hat{v}_{ic}(i_{ic}), \tag{2.7a}$$

$$v_{is} = -F_{is-q} \hat{v}_q(q) + F_{is-vs} v_{vs} - F_{is-ic} \hat{v}_{ic}(i_{ic}), \tag{2.7b}$$

$$v_{vc} = -F_{vc-q} \hat{v}_q(q) + F_{vc-vs} v_{vs} - F_{vc-ic} \hat{v}_{ic}(i_{ic}). \tag{2.7c}$$

Similarly, we can partition the currents as  $i = \text{col}(i_{\mathcal{L}}, i_{\mathcal{T}})$  with

$$i_{\mathcal{L}} = \begin{bmatrix} \hat{i}_{\phi}(\phi) \\ -i_{is} \\ \hat{i}_{vc}(v_{vc}) \end{bmatrix}, \quad i_{\mathcal{T}} = \begin{bmatrix} i_q \\ i_{vs} \\ i_{ic} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ i_{vs} \\ i_{ic} \end{bmatrix} \tag{2.8}$$

and write Kirchoff’s current law as  $i = B^{\top} i_{\mathcal{L}}$ . Simple matrix bookkeeping shows that

$$\dot{q} = F_{\phi-q}^{\top} \hat{i}_{\phi}(\phi) - F_{is-q}^{\top} i_{is} + F_{vc-q}^{\top} \hat{i}_{vc}(v_{vc}), \tag{2.9a}$$

$$i_{vs} = F_{\phi-vs}^{\top} \hat{i}_{\phi}(\phi) - F_{is-vs}^{\top} i_{is} + F_{vc-vs}^{\top} \hat{i}_{vc}(v_{vc}), \tag{2.9b}$$

$$i_{ic} = F_{\phi-ic}^{\top} \hat{i}_{\phi}(\phi) - F_{is-ic}^{\top} i_{is} + F_{vc-ic}^{\top} \hat{i}_{vc}(v_{vc}). \tag{2.9c}$$

Setting  $\mathcal{E} = \mathcal{E}_{\phi} + \mathcal{E}_q$  and noting that  $\hat{v}_q(q) = \nabla_q \mathcal{E}$  and  $\hat{i}_{\phi}(\phi) = \nabla_{\phi} \mathcal{E}$  we can write (2.7) and (2.9) as the port-Hamiltonian system [24]

$$\dot{x} = J \nabla_x \mathcal{E} + g_1 u_1 + g_2 u_2(y_2), \tag{2.10a}$$

$$y_1 = g_1^{\top} \nabla_x \mathcal{E} + h_{11} u_1 + h_{12} u_2(y_2), \tag{2.10b}$$

$$y_2 = -g_2^{\top} \nabla_x \mathcal{E} + h_{12}^{\top} u_1 + h_{22} u_2(y_2), \tag{2.10c}$$

with state  $x = \text{col}(\phi, q)$ , inputs  $u_1 = \text{col}(v_{vs}, i_{is})$  and  $u_2 = \text{col}(\hat{v}_{ic}(i_{ic}), \hat{i}_{vc}(v_{vc}))$ , outputs  $y_1 = \text{col}(i_{vs}, v_{is})$  and  $y_2 = \text{col}(i_{ic}, v_{vc})$ , and topological relationships

$$J = \begin{bmatrix} 0 & -F_{\phi-q} \\ F_{\phi-q}^\top & 0 \end{bmatrix}, \quad g_1 = \begin{bmatrix} F_{\phi-vs} & 0 \\ 0 & -F_{is-q}^\top \end{bmatrix}, \quad (2.11)$$

$$g_2 = \begin{bmatrix} -F_{\phi-ic} & 0 \\ 0 & F_{vc-q}^\top \end{bmatrix}, \quad h_{11} = \begin{bmatrix} 0 & -F_{is-vs}^\top \\ F_{is-vs} & 0 \end{bmatrix}, \quad (2.12)$$

$$h_{12} = \begin{bmatrix} 0 & F_{vc-vs}^\top \\ -F_{is-ic} & 0 \end{bmatrix}, \quad h_{22} = \begin{bmatrix} 0 & F_{vc-ic}^\top \\ -F_{vc-ic} & 0 \end{bmatrix}. \quad (2.13)$$

For ease of notation, whenever clear from the context, we drop the sub-index from the gradient operator, e.g., we write  $\nabla\mathcal{E}$  instead of  $\nabla_x\mathcal{E}$ .

A feature of the network representation given in (2.10) is that the power-flow relationship is clearly revealed. Indeed, the time derivative of the energy is

$$\begin{aligned} \dot{\mathcal{E}} &= \nabla\mathcal{E}^\top (J\nabla\mathcal{E} + g_1u_1 + g_2u_2) \\ &= \nabla\mathcal{E}^\top J\nabla\mathcal{E} + (y_1 - h_{11}u_1 - h_{12}u_2)^\top u_1 \\ &\quad - (y_2 - h_{12}^\top u_1 - h_{22}u_2)^\top u_2 \end{aligned} \quad (2.14)$$

so, by noting that  $J$ ,  $h_{11}$  and  $h_{22}$  are skew symmetric, we verify that

$$\dot{\mathcal{E}} = y_1^\top u_1 - y_2^\top u_2, \quad (2.15)$$

which shows that the rate at which the stored energy increases equals the difference between the power delivered by the sources and the power dissipated by the resistors.

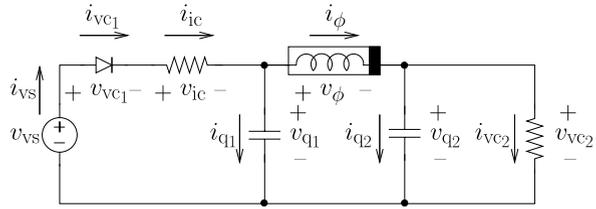
*Remark 2.4* Notice that for system (2.10) to be well defined it is necessary that a unique  $y_2$ —solution of (2.10c)—exists. In this paper we assume that such a  $y_2$  exists. The interested reader is referred to [19, 22], where sufficient conditions for existence and uniqueness can be found.

*Remark 2.5* If the characteristics of inductors and capacitors (2.1) are bijective, it is possible to relax Assumption 2.3 by finding a reduced equivalent network containing no inductor cut sets or capacitor loops. A precise notion of equivalence, as well as the explicit procedure to carry out the transformation, can be found in [21], see also [6]. Note, however, that in the general nonlinear case, practical use of these procedures is impeded by the requirement of an explicit solution of (2.10). An alternative way to relax Assumption 2.3 is to enforce Kirchhoff's laws using the notion of port-Hamiltonian models with constraints [15].

*Example* As an example, consider the circuit shown in Fig. 2. We model the diode as a nonlinear, voltage controlled resistor characterized by

$$i_{vc1} = I_{s1} \left( \exp\left(\frac{v_{vc1}}{V_T}\right) - 1 \right), \quad (2.16)$$

**Fig. 2** A nonlinear RLC circuit



where  $I_{s1}$  is the saturation current and  $V_T = 25$  mV. The other dissipative elements are a linear conductance governed by  $i_{vc2} = G v_{vc2}$  and the linear resistor  $v_{ic} = R i_{ic}$ . The nonlinear inductor, described by

$$i_{\phi} = I_{s\phi} \tanh\left(\frac{\phi}{\delta_{\phi}}\right), \tag{2.17}$$

saturates at a current  $I_{s\phi}$  and has, at the origin, an incremental inductance of  $\delta_{\phi}/I_{s\phi}$ . For simplicity, we consider linear capacitors of the form  $v_{q1} = q_1/C_1$  and  $v_{q2} = q_2/C_2$ .

The directed graph corresponding to the circuit is given in Fig. 3. The tree that satisfies Assumption 2.3 has been highlighted with thick lines.

The energy of the circuit is given by

$$\mathcal{E} = I_{s\phi} \delta_{\phi} \ln\left(\left|\cosh\left(\frac{\phi}{\delta_{\phi}}\right)\right|\right) + \frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2} + \mathcal{E}(0) \tag{2.18}$$

and its gradient by

$$\nabla \mathcal{E} = \begin{bmatrix} I_{s\phi} \tanh(\phi/\delta_{\phi}) \\ q_1/C_1 \\ q_2/C_2 \end{bmatrix}. \tag{2.19}$$

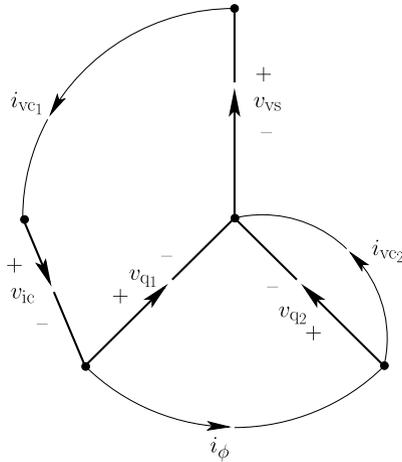
Recall that  $F = \{F_{\lambda-\tau}\}$ ,  $\lambda \in \mathcal{L}$ ,  $\tau \in \mathcal{T}$  is constructed according to the following rule:

$$F_{\lambda-\tau} = \begin{cases} 1 & \text{if } \tau \text{ is in the loop formed by } \lambda \\ & \text{their directions are equal,} \\ 0 & \text{if } \tau \text{ is not in the loop formed by } \lambda, \\ -1 & \text{if } \tau \text{ is in the loop formed by } \lambda \text{ but} \\ & \text{their directions are opposite.} \end{cases} \tag{2.20}$$

Since there are no current sources, the matrix  $F$  reduces to

$$F = \begin{bmatrix} F_{\phi-q} & F_{\phi-vs} & F_{\phi-ic} \\ F_{vc-q} & F_{vc-vs} & F_{vc-ic} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{2.21}$$

**Fig. 3** Directed graph corresponding to the circuit of Fig. 2. The *thick straight lines* represent the branches of  $\mathcal{T}$ . The *thin arcs* represent the links that belong to  $\mathcal{L}$



Finally, (2.10) becomes

$$\begin{bmatrix} \dot{\phi} \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \nabla \mathcal{E} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} u_2, \tag{2.22a}$$

$$i_{v_S} = [0 \quad 1 \quad 0] u_2, \tag{2.22b}$$

$$\begin{bmatrix} i_{i_C} \\ v_{v_{C1}} \\ v_{v_{C2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \nabla \mathcal{E} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v_{v_S} + \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u_2, \tag{2.22c}$$

where

$$u_2 = \hat{u}_2(y_2) = \begin{bmatrix} Ri_{i_C} \\ I_{S1}(\exp(v_{v_{C1}}/V_T) - 1) \\ Gv_{v_{C2}} \end{bmatrix}. \tag{2.23}$$

### 3 Relative Passivity and Global Output Regulation via PI Control

In this section, we study the problem of global output regulation of RLC networks described by (2.10). We will exploit their port-Hamiltonian structure to show that, under some suitable additional conditions on their characteristic functions, the problem can be solved with a simple PI controller.

It is well known [24] that output regulation to *zero* of systems of the form

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \tag{3.1}$$

is relatively simple if the map  $u \mapsto y$  is passive with a non-negative storage function; that is, if there exists a scalar function  $V(x) \geq 0$  such that  $\dot{V} \leq u^T y$ . In this case, a

simple proportional controller,  $u = -K_P y$  with  $K_P = K_P^\top > 0$ , ensures that (along all bounded trajectories)  $y(t) \rightarrow 0$ . Regulation will be global if the storage function is radially unbounded, which ensures boundedness of all trajectories.

It is widely known that RLC networks consisting of passive inductors, capacitors and resistors are passive [9]. This can be readily seen for circuits described by (2.10), using (2.15),  $y_2^\top u_2 \geq 0$ , and the fact that  $\mathcal{E}$  is non-negative for passive inductors and capacitors. In most practical applications of RLC circuits the control objective is not to drive the output to zero but to a desired value  $y_1^* \neq 0$ . In this case it is natural to look for passivity *relative* to  $y_1^*$  and its corresponding input [10]. More precisely, instead of looking for passivity of the map  $u_1 \mapsto y_1$ , we look for passivity of the map  $\tilde{u}_1 \mapsto \tilde{y}_1$ , where  $\tilde{u}_1 \triangleq u_1 - u_1^*$ ,  $\tilde{y}_1 \triangleq y_1 - y_1^*$  and  $u_1^*, x^*$  satisfy

$$0 = J \nabla \mathcal{E}^* + g_1 u_1^* + g_2 u_2(y_2^*), \tag{3.2a}$$

$$y_1^* = g_1 \nabla \mathcal{E}^* + h_{11} u_1^* + h_{12} u_2(y_2^*), \tag{3.2b}$$

$$y_2^* = -g_2 \nabla \mathcal{E}^* + h_{12}^\top u_1^* + h_{22} u_2(y_2^*) \tag{3.2c}$$

with  $\nabla \mathcal{E}^* \triangleq \nabla \mathcal{E}|_{x=x^*}$  for some  $y_2^*$ .

Passive linear systems are passive relative to any equilibrium input–output pair  $(u_1^*, y_1^*)$ , a property that is simply revealed by shifting the origin in the state space model (3.1) and noting that the storage function for the map  $u \mapsto y$  will also qualify as a storage function for  $\tilde{u} \mapsto \tilde{y}$ . Although this is in general not true for nonlinear systems [14], it follows from Assumption 2.3 that the voltage of each element of  $v_{\mathcal{L}}$  (in particular those of  $\dot{\phi}$ ) is a *linear* function of the voltages of  $v_{\mathcal{T}}$  (cf. (2.7a)). Likewise, the current of each element of  $i_{\mathcal{T}}$  (in particular those of  $\dot{q}$ ) is a linear function of the currents of  $i_{\mathcal{L}}$  (cf. (2.9a)). This known fact [7] is nicely captured in (2.10a) and suggests that, for this class of RLC networks, relative passivity holds. To prove it we need to strengthen Assumption 2.1, which was made for modeling purposes only.

**Assumption 3.1** The characteristics of the inductors and capacitors are strictly increasing and continuously differentiable. Those of the resistors are monotone non-decreasing. (Recall that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is monotone non-decreasing if  $(a - b)^\top (f(a) - f(b)) \geq 0, \forall a, b \in \mathbb{R}^n$ . It is strictly increasing if the inequality is strict whenever  $a \neq b$ .)

**Lemma 3.2** Under Assumption 3.1 the map  $\tilde{u}_1 \mapsto \tilde{y}_1$  is passive with the positive definite storage function

$$H(x) \triangleq \mathcal{E}(x) - (x - x^*)^\top \nabla \mathcal{E}^* - \mathcal{E}(x^*), \tag{3.3}$$

which is, furthermore, strictly convex.

*Proof* Since  $\nabla \mathcal{E} = \text{col}(\hat{i}_\phi(\phi), \hat{v}_q(q))$  and the inductors’ and capacitors’ characteristics are strictly increasing, we get that  $\mathcal{E}$  is strictly convex [13, p. 185]. It is immedi-

ately verified that

$$\begin{aligned} &H(a\lambda + (1 - \lambda)b) - [\lambda H(a) + (1 - \lambda)H(b)] \\ &= \mathcal{E}(a\lambda + (1 - \lambda)b) - [\lambda \mathcal{E}(a) + (1 - \lambda)\mathcal{E}(b)], \end{aligned} \tag{3.4}$$

which proves that strict convexity of  $\mathcal{E}$  is equivalent to strict convexity of  $H$ . Moreover, since  $\nabla H(x^*) = 0$ ,  $H(x^*) = 0$  is a unique global minimum and therefore  $H$  is positive definite (with respect to  $x^*$ ).

The network (2.10) can be written in terms of  $\tilde{u}$  and  $\tilde{y}$  as follows:

$$\begin{aligned} \dot{x} &= J\nabla\mathcal{E} + g_1u_1^* + g_1\tilde{u}_1 + g_2u_2^* + g_2\tilde{u}_2 \\ &= J(\nabla\mathcal{E} - \nabla\mathcal{E}^*) + g_1\tilde{u}_1 + g_2\tilde{u}_2, \end{aligned} \tag{3.5a}$$

$$\tilde{y}_1 = g_1^\top(\nabla\mathcal{E} - \nabla\mathcal{E}^*) + h_{11}\tilde{u}_1 + h_{12}\tilde{u}_2, \tag{3.5b}$$

$$\tilde{y}_2 = -g_2^\top(\nabla\mathcal{E} - \nabla\mathcal{E}^*) + h_{12}^\top\tilde{u}_1 + h_{22}\tilde{u}_2, \tag{3.5c}$$

where the second line is due to (3.2a). The derivative of  $H$  is then obtained as before:

$$\begin{aligned} \dot{H} &= (\nabla\mathcal{E} - \nabla\mathcal{E}^*)^\top(J(\nabla\mathcal{E} - \nabla\mathcal{E}^*) + g_1\tilde{u}_1 + g_2\tilde{u}_2) \\ &= \tilde{y}_1^\top\tilde{u}_1 - \tilde{y}_2^\top\tilde{u}_2. \end{aligned} \tag{3.6}$$

Finally, the monotonicity of the resistors leads to  $\dot{H} \leq \tilde{y}_1^\top\tilde{u}_1$ . □

*Remark 3.3* Strict convexity of  $\mathcal{E}$  might seem too strong if all that we need is an  $H$  bounded from below. The reason for imposing strict convexity is that, together with the existence of a minimum, it implies radial unboundedness [14]. This will prove useful for global stabilization later on.

As indicated above, the output of a passive system can be regulated to zero with a proportional feedback. In the present context, this evokes a control law of the form  $\tilde{u}_1 = -K_P\tilde{y}_1$ , whose implementation  $u_1 = -K_P\tilde{y} + u_1^*$  clearly requires the exact value of the feed-through term  $u_1^*$ . The latter—obtained from the solution of (3.2)—requires a precise knowledge of the system parameters, rendering the controller highly non-robust. This problem can be surmounted by the use of an integral action, as shown in the following theorem.

**Theorem 3.4** Consider network (2.10). Under Assumption 3.1 the PI controller

$$\dot{\xi} = -\tilde{y}_1, \tag{3.7a}$$

$$u_1 = K_I\xi - K_P\tilde{y}_1 \tag{3.7b}$$

with  $K_I = K_I^\top > 0$  and  $K_P = K_P^\top > 0$  ensures that for all initial conditions,  $y_1(t)$  converges asymptotically to  $y_1^*$ , that is,

$$\lim_{t \rightarrow \infty} y_1(t) = y_1^*. \tag{3.8}$$

Moreover, it globally stabilizes the equilibrium point  $(x^*, \xi^* \triangleq K_1^{-1}u^*)$ . (By global stability we mean stability in the sense of Lyapunov plus boundedness of the solutions for every initial condition.) If in addition, the closed-loop system (2.10), (3.7) satisfies the detectability condition

$$\tilde{y}_1(t) \equiv 0 \implies \lim_{t \rightarrow \infty} (x(t), \xi(t)) = (x^*, \xi^*), \tag{3.9}$$

then the equilibrium is globally asymptotically stable.

*Proof* It is straightforward to see that the (shifted) candidate Lyapunov function

$$V(x, \xi) = H(x) + \frac{1}{2}(\xi - \xi^*)^\top K_1(\xi - \xi^*) \tag{3.10}$$

is indeed a Lyapunov function:

$$\begin{aligned} \dot{V} &= \dot{H} + \dot{\xi}^\top K_1(\xi - \xi^*) \\ &= \tilde{y}_1^\top \tilde{u}_1 - \tilde{y}_2^\top \tilde{u}_2 - \tilde{y}_1^\top K_1(\xi - \xi^*) \\ &= \tilde{y}_1^\top \tilde{u}_1 - \tilde{y}_2^\top \tilde{u}_2 - \tilde{y}_1^\top (\tilde{u}_1 + K_P \tilde{y}_1) \\ &= -\tilde{y}_2^\top \tilde{u}_2 - \tilde{y}_1^\top K_I \tilde{y}_1 \leq 0. \end{aligned} \tag{3.11}$$

The second equation is due to (3.6) and (3.7a), while the third is due to (3.7b) and the definition of  $\xi^*$ . Non-positivity is due to the monotonicity of the resistors and the positive definiteness of  $K_1$ . It follows then from standard Lyapunov theory that  $(x^*, \xi^*)$  is stable. Since  $V$  is also radially unbounded (see Remark 3.3) the solutions remain bounded for any initial condition [16, p. 124]. From LaSalle’s invariance principle [20, p. 66] we conclude that  $y_1(t)$  converges to  $y_1^*$  and that the detectability condition (3.9) leads to asymptotic stability.  $\square$

*Example (cont.)* The derivatives of the inductor’s and capacitors’ characteristics are, respectively,

$$\frac{I_{s\phi}}{\delta\phi} \operatorname{sech}^2\left(\frac{\phi}{\delta\phi}\right), \quad \frac{1}{C_1} \quad \text{and} \quad \frac{1}{C_2}.$$

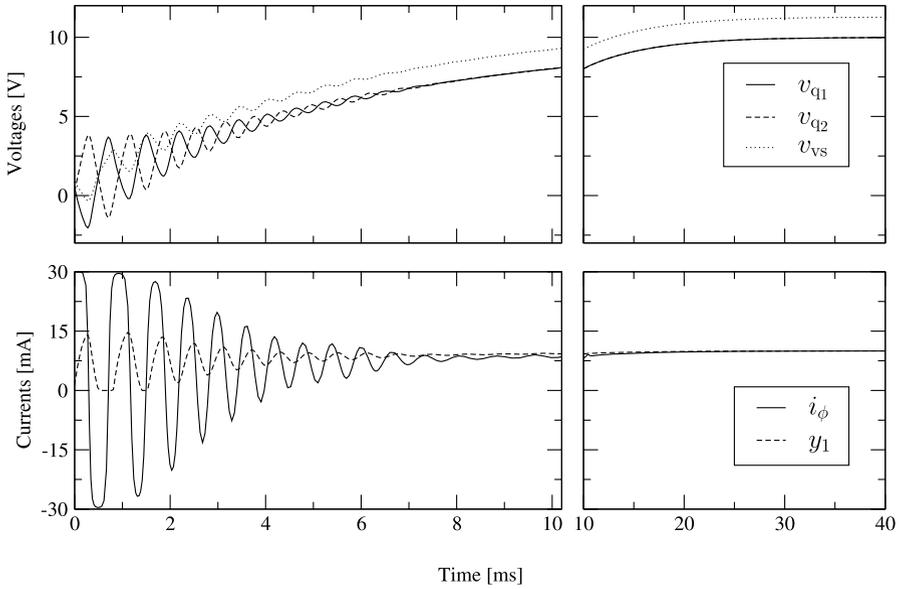
Those of the resistive elements are

$$R, \quad \frac{I_{s1}}{V_T} \exp\left(\frac{v_{vc1}}{V_T}\right) \quad \text{and} \quad G.$$

Assumption 3.1 is verified, since strict positivity of the derivatives implies strict monotonicity.

In order to obtain the equilibrium input–output pairs, it is necessary to solve (3.2):

$$y_2^* = \begin{bmatrix} i_{ic}^* \\ v_{vc1}^* \\ v_{vc2}^* \end{bmatrix} = \begin{bmatrix} y_1^* \\ V_T \ln(y_1^*/I_{s1} + 1) \\ y_1^*/G \end{bmatrix}, \tag{3.12}$$



**Fig. 4** Simulated response of the regulated circuit

$$x^* = \begin{bmatrix} \phi^* \\ q_1^* \\ q_2^* \end{bmatrix} = \begin{bmatrix} \delta_\phi \operatorname{artanh}(y_1^*/I_{s\phi}) \\ y_1^* C_1 / G \\ y_2^* C_2 / G \end{bmatrix}, \tag{3.13}$$

$$u_1^* = v_{vs} = v_{vc1}^* + y_1^* \left( \frac{1}{G} + R \right). \tag{3.14}$$

Suppose that the system parameters are

$$I_{s\phi} = 30 \text{ mA}, \quad \delta_\phi = 250 \text{ }\mu\text{Wb}, \quad C_1 = C_2 = 2 \text{ }\mu\text{F} \tag{3.15}$$

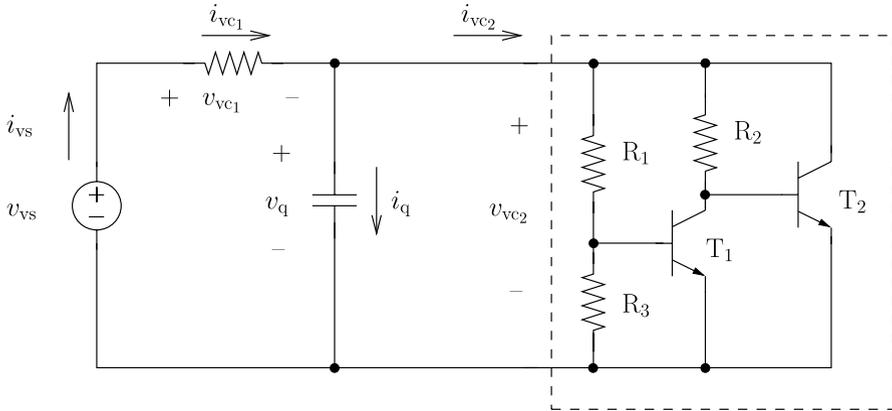
and

$$R = 100 \text{ }\Omega, \quad I_{s1} = 0.1 \text{ }\mu\text{A}, \quad G = 1 \text{ mA/V}. \tag{3.16}$$

Suppose further that we want to stabilize the diode’s current at  $y_1^* = 10 \text{ mA}$ . For an initial condition

$$x(0) = \begin{bmatrix} 1 \text{ mWb} \\ 1 \text{ }\mu\text{C} \\ 0.5 \text{ }\mu\text{C} \end{bmatrix}, \tag{3.17}$$

the PI controller (3.7), with  $K_I = 500 \text{ V/mA}\cdot\text{s}$  and  $K_P = 100 \text{ }\Omega$ , produces the set of currents and voltages shown in Fig. 4. It can be seen that both  $y_1$  and  $u_1$  converge to their preset values of 10 mA and 11.3 V respectively.



**Fig. 5** A nonlinear RLC circuit with a non-monotonic resistance

*Counter example* To underscore the importance of the monotonicity of the resistors, consider the circuit shown in Fig. 5. The components within the dashed box are modeled as a single voltage controlled resistive port. The remaining resistor is also considered to be voltage controlled. The graph of the circuit is thus partitioned as

$$v_{\mathcal{T}} = \begin{bmatrix} v_q \\ v_{vs} \end{bmatrix}, \quad v_{\mathcal{L}} = \begin{bmatrix} v_{vc1} \\ v_{vc2} \end{bmatrix} \tag{3.18}$$

and the equations of the circuit are

$$\dot{q} = \hat{i}_{vc1}(v_{vc1}) - \hat{i}_{vc2}(v_{vc2}), \tag{3.19a}$$

$$i_{vs} = \hat{i}_{vc1}(v_{vc1}), \tag{3.19b}$$

$$\begin{bmatrix} v_{vc1} \\ v_{vc2} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \hat{v}_q(q) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_{vs}. \tag{3.19c}$$

The single resistor and the capacitor are assumed to be linear with characteristics

$$i_{vc1} = Gv_{vc1} \quad \text{and} \quad v_q = \frac{1}{C}q. \tag{3.20}$$

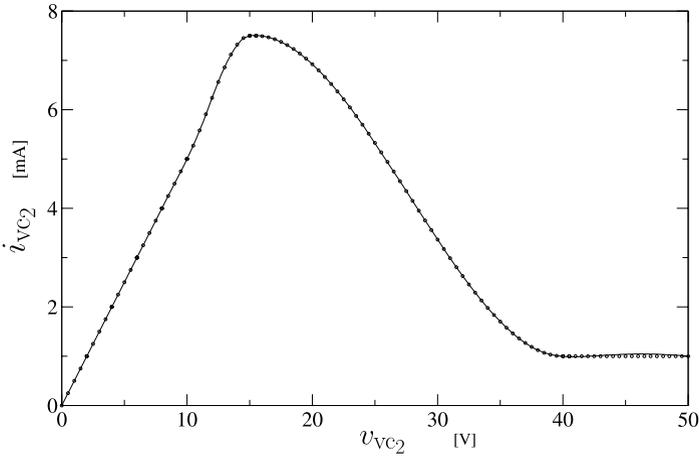
According to [23], the nonlinear port exhibits an N-type negative differential resistance (NDR), like the one shown in Fig. 6.

Suppose that

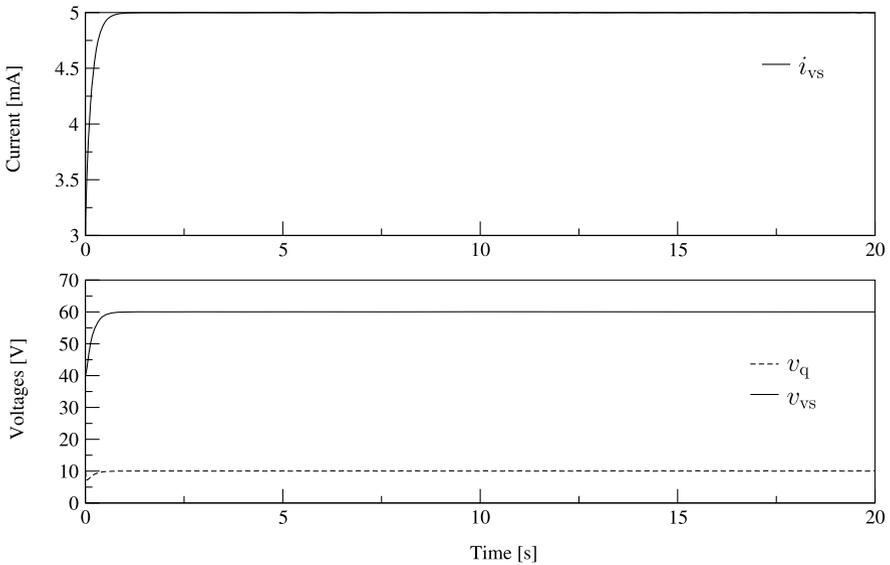
$$G = 10 \text{ mA/V}, \quad C = 10 \text{ }\mu\text{F} \tag{3.21}$$

and that the transistors and resistors are such that measurements produce the set of points shown in Fig. 6. Suppose further that we want to set the output current  $y_1^* = i_{vs}^*$  at 5 mA, so we construct the controller (3.7) with

$$K_P = 2 \frac{1}{G} \quad \text{and} \quad K_I = 2C. \tag{3.22}$$

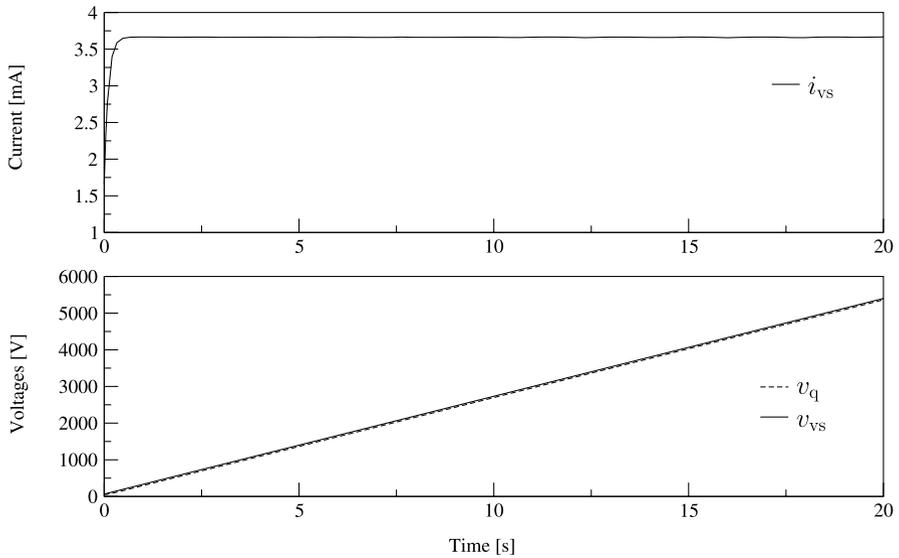


**Fig. 6** Characteristic of the N-type negative differential resistance



**Fig. 7** Simulated response with initial condition  $\hat{v}_q(q(0)) = 10$  V

For an initial condition of 0.1 mC, the voltage of the capacitor equals 10 V. Since at this voltage the differential resistance is positive, we can expect the state of the circuit to converge to the desired value (see Fig. 7). On the other hand, an initial condition of 0.5 mC sets a voltage of the capacitor which is within the NDR zone. It can be seen (Fig. 8) that the output of the circuit does not converge to the desired value. Moreover, the state and the control diverge. Thus, the closed-loop circuit is locally stable, but not globally.



**Fig. 8** Simulated response with initial condition  $\hat{v}_q(q(0)) = 50$  V

## 4 Conclusions

We have identified in this paper a large class of nonlinear RLC circuits that can be (globally) stabilized with simple PI controllers. Instrumental for our proof was the establishment of the property of relative passivity, which is satisfied by RLC circuits with monotonic characteristic functions. It has been shown that the incorporation of the integral action robustifies the controller, obviating the need for an exact knowledge of the circuit parameters. This well-known property of PI controllers—which underlies its huge popularity in applications—is particularly critical where the implementation of a purely proportional control action requires the computation of the constant input associated to a desired equilibrium. Current research is under way to explore the use of relative passivity to induce oscillations on the circuit, a behavior that is desired in many practical problems.

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