Implicit IDA-PBC for Underactuated Mechanical Systems: An LMI-based Approach

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Abstract— **Recently, the Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC) methodology has been extended to underactuated mechanical systems in implicit port-Hamiltonian representation. The method is not restricted to holonomic systems, does not require a positive-definite target inertia matrix and, under general conditions, avoids the need for solving partial differential equations. In this paper we simplify the conditions for (local) stability and present equivalent matching equations. In addition, we exploit the inherent polynomial structure of implicit systems modeled in Euclidean space, such that the implicit IDA-PBC problem can be cast as a linear matrix inequality (LMI) problem. The method is applicable to desired Hamiltonians with arbitrary polynomial order. The proposed methodology is validated on the portal crane and the cart-pole system.**

I. Introduction

At least two different representations can be used when modeling port-Hamiltonian systems [1]: (1) the implicit representation, where system models are obtained by aggregating simpler subsystems and the dynamics are described by differential-algebraic equations with the interconnections expressed as algebraic constraints, and (2), the explicit representation, where the interconnections are simplified and the system is handled as a whole. In the latter, ordinary differential equations represent the dynamics. In the former implicit mechanical framework, constraints may be interpreted, e.g., as interconnections (joints) between rigid body (links) elements.

In this context, the Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC) has been intensively studied on explicit systems, capturing a wide range of applications, see e.g. [2], [3], [4], [5], [6]. In contrast, only few research has been devoted to IDA-PBC in implicit systems. The first work started with A. Macchelli [7], where a general IDA-PBC approach is introduced for implicit port-Hamiltonian systems, with application focused on discretized infinite-dimensional systems. Respective conditions are stated in image and kernel Dirac representations with linear maps. Later, the authors of [8] take a different perspective. They focus on underactuated mechanical systems (UMS)s with a representation equivalent to a Dirac structure given by a combination of hybrid and constrained input-output representations, see [9]. Their algorithm does not modify the interconnection and dissipation matrix, but, for a class of holonomic systems, is able to reduce the matching conditions to a simple quadratic programming problem.

Only recently, Cieza and Reger [10] have introduced a total energy shaping IDA-PBC for UMSs in implicit port-Hamiltonian representation. The general method is not restricted to holonomic systems, does not require a positive definite target inertia matrix, and under certain conditions (holonomy of the constraints) can reduce the kinetic and potential partial differential equations to algebraic equations. Nonetheless, among the main restrictions, it requires a constant (closed-loop) target inertia matrix, a specific form for the desired potential energy and, depending on the system, solving their algebraic equations may still turn out to be a difficult task.

On the other hand, the celebrated book of Boyd et al. [11] has laid open the wide range of control problems that can be stated as linear matrix inequalities (LMI)s. Most of the theory was focused first on linear systems with non-linearities modeled as uncertainty [12]. The contribution of LMIs in nonlinear control roughly started with the sum of squares (SOS) approach [13] and successfully led to the synthesis of robust, optimal controllers for polynomial systems in explicit representation [14], [15]. The design of such controllers for UMSs is however quite recent and most results remain local and with a small application spectrum [16], [17], [18].

In light of this, mechanical systems modeled implicitly in Euclidean space possess a typical polynomial structure which naturally leads to the question: Can we develop implicit controllers for UMSs using SOS (or LMI) solvers? In this work we answer this question affirmatively by first relaxing the stability conditions of [10] and presenting equivalent matching equations such that the problem is expressible in terms of LMIs with arbitrary polynomial order in the desired Hamiltonian, that is, in the target inertia matrix and potential energy, extending the algorithm of [10].

The paper is structured as follows. In Section II we recall the concepts of IDA-PBC for underactuated mechanical systems in implicit port-Hamiltonian representation. Section III presents a relaxation of the stability conditions and equivalent matching equations. In Section IV we discuss the parameter selection that allows the LMI implementation. We verify our results in Section V on two example systems: a cart-pole and a portal crane. We draw the conclusions in Section VI.

II. Background

Let us summarize the general formulation of the total energy shaping IDA-PBC for UMSs in implicit port Hamil-

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tonian representation as presented in [10]. Consider the nominal system[1]

$$\begin{bmatrix}\dot r\\ \dot\rho\end{bmatrix} = \begin{bmatrix}0 & I_{n_r}\\ -I_{n_r} & 0\end{bmatrix}\begin{bmatrix}\partial_r^\top\mathcal{H}\\ \partial_\rho^\top\mathcal{H}\end{bmatrix} + \begin{bmatrix}0\\ b(r)\end{bmatrix}\lambda + \begin{bmatrix}0\\ \mathcal{G}(r)\end{bmatrix}u, \tag{1a}$$

$$0 = b^\top(r)\partial_\rho^\top\mathcal{H}, \tag{1b}$$

$$\mathcal{H}(r,\rho) = \mathcal{V}(r) + \frac{1}{2}\rho^\top\mathcal{M}^{-1}(r)\rho,$$

and the desired (or target) system

$$\begin{bmatrix}\dot r\\ \dot\rho\end{bmatrix} = \begin{bmatrix}0 & \mathcal{J}(r)\\ -\mathcal{J}^\top(r) & -\mathcal{W}(r,\rho)\end{bmatrix}\begin{bmatrix}\partial_r^\top\mathcal{H}_d\\ \partial_\rho^\top\mathcal{H}_d\end{bmatrix} + \begin{bmatrix}0\\ b_d(r)\end{bmatrix}\lambda_d, \tag{2a}$$

$$0 = b_d^\top(r)\partial_\rho^\top\mathcal{H}_d, \tag{2b}$$

$$\mathcal{H}_d(r,\rho) = \mathcal{V}_d(r) + \frac{1}{2}\rho^\top\mathcal{M}_d^{-1}(r)\rho,$$

where $r\in\mathbb{R}^{n_r}$ and $\rho\in\mathbb{R}^{n_r}$ are implicit generalized coordinates (positions) and momenta, $u\in\mathbb{R}^{n_u}$ is the input, $\mathcal{G}:\mathbb{R}^{n_r}\to\mathbb{R}^{n_r\times n_u}$ is the implicit full rank input matrix, $b(r)\lambda$ and $b_d(r)\lambda_d$ represent the constraint forces with $b,\ b_d:\mathbb{R}^{n_r}\to\mathbb{R}^{n_r\times n_\lambda}$ and $\lambda,\ \lambda_d\in\mathbb{R}^{n_\lambda}$ are the implicit variables. $\mathcal{M},\ \mathcal{M}_d:\mathbb{R}^{n_r}\to\mathbb{R}^{n_r\times n_r}$ are symmetric inertia matrices with $\mathcal{M}\succ 0$. The Hamiltonian $\mathcal{H}:\mathbb{R}^{n_r}\times\mathbb{R}^{n_r}\to\mathbb{R}$ gives the total energy (potential plus kinetic) and $\mathcal{H}_d$ is the desired energy function. To avoid cumbersome notation, we will omit the arguments of functions that have been previously defined.

Proposition 1 (Well posedness): Consider the implicit system (1) and define

$$\mathcal{X} := \{r\in\mathbb{R}^{n_r} \mid \text{rank }\Delta(r) = n_\lambda\},\quad \Delta(r) := b^\top\mathcal{M}^{-1}b.$$

Then, for any state with $r\in\mathcal{X}$ there exists a unique solution for λ. In addition, the constraint manifold $\mathcal{X}_c = \left\{(r,\rho)\in\mathcal{X}\times\mathbb{R}^{n_r} \mid b^\top\partial_\rho^\top\mathcal{H} = 0\right\}$ is well defined and the DAE system (1) has differential index 1.

Proof: See [9]. ■

In a similar way, we can define $\Delta_d(r) := b_d^\top\mathcal{M}_d^{-1}b_d$, $\mathcal{X}_d := \{r\in\mathbb{R}^{n_r} \mid \text{rank }\Delta_d(r) = n_\lambda\}$, such that the target system (2) is well defined for all $r\in\mathcal{X}_d$.

Proposition 2 (Matching equations [10]): System (1) can be transformed into (2) for any trajectory of r that remains in $\mathcal{X}\cap\mathcal{X}_d$ whenever the following Kinetic (quadratic in ρ)

$$S_\perp\left(\partial_r^\top(\mathcal{M}^{-1}\rho) - \mathcal{J}^\top\partial_r^\top(\mathcal{M}_d^{-1}\rho) - \mathcal{W}_1\mathcal{M}_d^{-1}\right)\rho = 0, \tag{3a}$$

Potential (independent of ρ)

$$S_\perp\left(\partial_r^\top\mathcal{V} - \mathcal{J}^\top\partial_r^\top\mathcal{V}_d\right) = 0 \tag{3b}$$

and constraint

$$S_\perp\mathcal{J}^\top b = 0 \tag{3c}$$

matching conditions are satisfied. Consequently, the uniquely defined control law is given by

$$u_I = \begin{bmatrix}I_m & 0\end{bmatrix}(S^\top S)^{-1}S^\top\left(\partial_r^\top\mathcal{H} + L + \mathcal{J}^\top b\lambda_d\right), \tag{4}$$

where $L = -\mathcal{J}^\top\partial_r^\top\mathcal{H}_d - \mathcal{W}\partial_\rho^\top\mathcal{H}_d$, $\mathcal{J} = \mathcal{M}^{-1}\mathcal{M}_d$, $b_d = \mathcal{J}^\top b$, $S = \begin{bmatrix}\mathcal{G} & b\end{bmatrix}$, $S_\perp$ is the full rank left annihilator[2] of S,

$$\mathcal{W}(r,\rho) = S(r)K_u(r)S^\top(r) + \frac{1}{2}\mathcal{W}_1(r,\rho),$$

$K_u(r)\in\mathbb{R}^{(n_u+n_\lambda)\times(n_u+n_\lambda)}$, and $\mathcal{W}_1$ is affine in ρ.

Remark 1: The constraint (1b) is a physical property of the system. Therefore, (2b) is only an equivalent representation of (1b). The matching condition (3c), which introduces conservativeness, allows to solve (3a) and (3b) independently of the solution of λ_d, which would otherwise increase the problem complexity.

Remark 2: If desired, the implicit variables λ and λ_d can be calculated using the hidden (or secondary) constraint $\frac{d\, b^\top\partial_\rho^\top\mathcal{H}}{dt} = \partial_r\left(b^\top\partial_\rho^\top\mathcal{H}\right)\partial_\rho^\top\mathcal{H} + b^\top\mathcal{M}^{-1}\dot\rho = 0$.

Proposition 3 (Implicit stability [10]): Assume that the conditions of Prop. 2 are satisfied and define the new domain $\mathcal{X}_I = \left(\left\{r\in\mathcal{X}_d \mid b_\perp\bar{\mathcal{M}}_d b_\perp^\top\succ 0, \Phi = 0\right\}\times\mathbb{R}^{n_r}\right)\cap\mathcal{X}_c$. Then $x^\star = (r^\star, 0)\in\mathcal{X}_a = \left\{(r,\rho)\in\mathcal{X}_I \mid S_\perp\partial_r^\top\mathcal{V} = 0\right\}$ is a stable equilibrium of closed-loop (2) for any $K_u + K_u^\top\succeq 0$ if [3]

$$x^\star = \arg\min\ \mathcal{V}_d|_{\mathcal{X}_I} \tag{5a}$$

is an isolated minimum and

$$0 = \rho^\top\mathcal{M}_d^{-1}\mathcal{W}_1\mathcal{M}_d^{-1}\rho\big|_{\mathcal{X}_I}. \tag{5b}$$

Furthermore, if $y_I := (K_u + K_u^\top)^{\frac{1}{2}}S^\top\mathcal{M}_d^{-1}\rho$ is a detectable output of (2), $x^\star$ is asymptotically stable.[4] Here $\bar{\mathcal{M}}_d = \mathcal{M}\mathcal{M}_d^{-1}\mathcal{M}$, $\Phi(r) := \int_0^r\bar b^\top(s)ds + c$ are the integrated constraints,[5] $\partial_r\Phi\equiv\bar b^\top(r)$, c is constant, $b^\top\partial_\rho^\top\mathcal{H}\in\mathcal{C}^1$ and $\bar b$ is matrix constructed with horizontal concatenation of the columns of b, namely b_i, satisfying the integrability condition $\partial_r b_i\equiv\partial_r^\top b_i$.

Propositions 1–3 summarize the implicit IDA-PBC methodology. The set $\mathcal{X}_I$ contains the region of convergence.

III. Main Results

In this section we present simpler conditions that are locally sufficient for those of Proposition 3. We then introduce equivalent matching equations to further simplify the analysis.

Proposition 4: Given the implicit system (1) with desired position $r^\star\in\left\{r\in\mathcal{X} \mid S_\perp\partial_r^\top\mathcal{V} = 0\right\}$, there exist a (locally) stabilizing control law (4) in $(r^\star, 0)$ whenever:

i) $\mathcal{M}_d(r^\star)$ has full rank.
ii) $K_u + K_u^\top\succeq 0$.
iii) $\mathcal{M}_d$, $\mathcal{V}_d$ and $\mathcal{W}_1$ satisfy the matching conditions (3) with (5b) and

$$b_\perp\mathcal{M}\mathcal{M}_d^{-1}\mathcal{M}b_\perp{}^\top\Big|_{r=r^\star}\succ 0. \tag{6a}$$

[1] We write $\frac{\partial h}{\partial x} = \partial_x h$ or $\left(\frac{\partial h}{\partial x}\right)^\top = \partial_x^\top h$ for any vector or scalar function $h(x)$. I_n is the identity matrix of size n.

[2] System (1) is underactuated if rank $S < n_r$, see [8]. Consequently, $S_\perp$ exists, i.e., (3) plays a role, if and only if (1) is underactuated.

[3] Note that $\mathcal{H}_d|_{\mathcal{X}_I}$ is the restriction of $\mathcal{H}_d$ to $\mathcal{X}_I$.

[4] We write ${A^{\frac{1}{2}}}^\top A^{\frac{1}{2}} = A$ for any square positive semidefinite matrix A.

[5] If all constraints are non-integrable, i.e., non-holonomic, there is no Φ and the domain $\mathcal{X}_I$ reduces to $\left(\left\{r\in\mathcal{X}_d \mid b_\perp\bar{\mathcal{M}}_d b_\perp^\top\succ 0\right\}\times\mathbb{R}^{n_r}\right)\cap\mathcal{X}_c$.

iv) There is some constant $\mu^\star \in \mathbb{R}^{n_\lambda}$ such that[6]

$$\bar{b}_\perp \left(\partial_r \left(\bar{b}\mu^\star \right) + \frac{\partial^2 \mathcal{V}_d}{\partial r^2} \right) \bar{b}_\perp^\top \bigg|_{r=r^\star} \succ 0 \quad \text{and} \tag{6b}$$

$$\partial_r \mathcal{V}_d(r^\star) + \bar{b}(r^\star)\mu^\star = 0, \tag{6c}$$

with $\bar{b}_\perp$ the full rank left annihilator of $\bar{b}$.

Proof: Define $\mathcal{X}_{s1} = \left\{ r \in \mathcal{X} \mid b_\perp \bar{\mathcal{M}}_d b_\perp^\top \succ 0 \right\}$ and use

$$\begin{bmatrix} b_\perp \\ b^\top \end{bmatrix} \begin{bmatrix} \bar{\mathcal{M}}_d b_\perp^\top & b \end{bmatrix} = \begin{bmatrix} b_\perp \bar{\mathcal{M}}_d b_\perp^\top & 0 \\ b^\top \bar{\mathcal{M}}_d b_\perp^\top & b^\top b \end{bmatrix},$$

then $\begin{bmatrix} \bar{\mathcal{M}}_d b_\perp^\top & b \end{bmatrix}$ has full rank for all $r \in \mathcal{X}_{s1}$. If $\mathcal{M}_d(r^\star)$ has full rank, there also exists a neighborhood $\mathcal{X}_{s2}$ of $r^\star$ where $\mathcal{M}_d$ has full rank. Now, from the identity

$$\begin{bmatrix} b_\perp \\ b^\top \end{bmatrix} \bar{\mathcal{M}}_d^{-1} \begin{bmatrix} \bar{\mathcal{M}}_d b_\perp^\top & b \end{bmatrix} = \begin{bmatrix} b_\perp b_\perp^\top & b_\perp \bar{\mathcal{M}}_d b \\ 0 & \Delta_d \end{bmatrix},$$

we note that Δ_d has full rank for all $r \in \mathcal{X}_{s1} \cap \mathcal{X}_{s2}$. Therefore, $(\mathcal{X}_{s1} \cap \mathcal{X}_{s2}) \subset \mathcal{X}_I$, i.e., $(r^\star, 0) \in \mathcal{X}_I$. Using Prop. 2 and 3, it remains to show that, locally, $(r^\star, 0) = \arg\min \mathcal{H}_d|_{\mathcal{X}_I}$. For this, assume that at least one constraint in (1b) is integrable and define the Lagrangian function $\mathcal{L}_d(r, \rho, \nu, \mu) = \mathcal{H}_d + \nu^\top b^\top \mathcal{M}^{-1}\rho + \mu^\top \Phi$, with Lagrange multipliers μ and ν, and constraints $b^\top \mathcal{M}^{-1}\rho = 0$ and $\Phi = 0$. The necessary and sufficient conditions, see [19], for $x^\star$ to be a strict local minimum of $\mathcal{H}_d|_{\mathcal{X}_b}$ are given by

$$\partial_x^\top \mathcal{L}_d(r^\star, \rho^\star, \nu^\star, \mu^\star) = 0,$$

$$\partial_x^\top \mathcal{L}_d = \begin{bmatrix} \partial_r^\top \mathcal{L}_d \\ \partial_\rho^\top \mathcal{L}_d \end{bmatrix} = \begin{bmatrix} \partial_r^\top \mathcal{V}_d + \bar{b}\mu + \partial_r^\top\big(\rho^\top \mathcal{M}^{-1} b\nu\big) \\ \mathcal{M}^{-1} b\nu + \mathcal{M}_d^{-1}\rho \end{bmatrix}.$$

$$y_\rho^\top \mathcal{M}_d^{-1} y_\rho > 0, \quad y_\rho^\top \mathcal{M}^{-1} b^\star = 0, \quad \text{and} \tag{7a}$$

$$y_r^\top \left(\frac{\partial^2 \mathcal{V}_d}{\partial r^2} + \partial_r \bar{b}\mu^\star \right) \bigg|_{r=r^\star} y_r > 0, \quad y_r^\top \bar{b}(r^\star) = 0. \tag{7b}$$

From $\partial_x^\top \mathcal{L}_d\big|_{x=x^\star} = 0$ and the full rank condition of Δ_d we obtain $\nu^\star = 0$ and (6c). Then we employ Finsler's Lemma on (7a) and (7b) which results in (6a) and (6b), respectively. In case of non-holonomic systems, $\bar{b}$ and μ do no exist; consequently, $\bar{b}_\perp$ can be selected as I_{n_r}, and $\bar{b}\mu$ as 0. ∎

Proposition 4 simplifies Prop. 3 in the following way:

- It is not required to search for sets $\mathcal{X}_d$ or $\mathcal{X}_I$ to guarantee the existence of a control law.
- Instead of searching for $x^\star = \arg\min \mathcal{H}_d|_{\mathcal{X}_I}$ in the whole $\mathcal{X}_I$, we only need to search locally (in $r^\star$) for (6) with full rank $\mathcal{M}_d$

The implicit IDA-PBC stated in Prop. 4 does not require $\mathcal{M}_d(r)$ to be strictly positive definite, only nonsingular ($\mathcal{M}_d$ must be invertible) with $b_\perp \bar{\mathcal{M}}_d b_\perp^\top \succ 0$. This weaker condition allows, e.g., the existence of a local controller for the cart-pole with $\mathcal{M}_d$ constant, which otherwise would be impossible by the condition $\mathcal{M}_d \succ 0$, see [10]. However, analyzing the elements of $\mathcal{M}_d$, such that it fulfills $b_\perp \bar{\mathcal{M}}_d b_\perp^\top \succ 0$ and the matching conditions (3), is not always easy (even for constant $\mathcal{M}_d$) due to the matrix inversion. The following proposition aims at simplifying this problem.

[6] If all constraints (1b) are non integrable, then (6b) reduces to $\frac{\partial^2 \mathcal{V}_d}{\partial r^2} \succ 0$ and (6c) is equivalent to $\partial_r \mathcal{V}_d(r^\star) = 0$.

Proposition 5 (Equivalent Matching): The matching conditions (3) are equivalent to

$$k b_\perp \big(\mathcal{M}\mathcal{M}_d^{-1} \partial_r^\top(\mathcal{M}^{-1}\rho) - \bar{\mathcal{W}}_1 - \partial_r^\top(\mathcal{M}_d^{-1}\rho)\big)\rho = 0 \tag{8a}$$

$$k b_\perp \left(\mathcal{M}\mathcal{M}_d^{-1} \partial_r^\top \mathcal{V} - \partial_r^\top \mathcal{V}_d\right) = 0 \tag{8b}$$

$$k b_\perp \mathcal{M}\mathcal{M}_d^{-1} S = 0 \tag{8c}$$

for all $(r, \rho) \in (\mathcal{X} \cap \mathcal{X}_d) \times \mathbb{R}^{n_r}$, where $\bar{\mathcal{W}}_1 = \mathcal{M}\mathcal{M}_d^{-1}\mathcal{W}_1\mathcal{M}_d^{-1}$, $b_\perp$ is the full rank left annihilator of b and $k(r) \in \mathbb{R}^{(n_r - n_u - n_\lambda)\times n_r}$ is a full rank matrix.

Proof: Using (8c) and the full rank condition on k, we obtain

$$k(r) b_\perp = \bar{k}(r) S_\perp \mathcal{M}_d \mathcal{M}^{-1} \tag{9}$$

for a square full rank matrix $\bar{k}(r)$. Replacing (9) in (8) and multiplying by $\bar{k}^{-1}$ yields (3). The necessity proof follows a similar procedure but with (3c). ∎

Inspection of the equivalent matching equations (8) and the conditions (6) of Prop. 4 shows that only $\mathcal{M}_d^{-1}$ is present ($\mathcal{M}_d$ has been removed), avoiding in this way the inversion problem at the cost of introducing the new unknown matrix $k(r)$. This allow us to state the problem as an LMI problem, shown in Section IV-A.

IV. Heuristic Solutions

A. Implicit IDA-PBC using LMI solvers

Assume $\mathcal{W}_1$, $\mathcal{M}_d^{-1}$ and $\mathcal{V}_d$ are given by linear combinations of known basis functions. Then, using Prop. 4 we can express the matching conditions (3) and (6c) as $F_1(x, \gamma) = 0$ and express (6a)–(6b) as $F_0(x^\star, \gamma) \succ 0$, where $x = (r, \rho)$, γ is a vector of the unknown coefficients, F_1 is a nonlinear function in x and γ, and F_0 is affine in γ.

Typically, implicit mechanical systems that are modeled in Euclidean space have constant $\mathcal{M}$ and polynomial $\mathcal{V}$ and $\mathcal{S}$, see [10], [8].[7] Therefore, instead of searching for k and $\mathcal{M}_d^{-1}$ in the equivalent matching conditions (8) simultaneously, we fix k and search for polynomial $\mathcal{N}_i$, $\mathcal{M}_d^{-1}$ and $\mathcal{V}_d$ with a specific structure in $\mathcal{W}_1$ parametrized by $\mathcal{N}_i$, see Section IV-C. Consequently, we can express (8) as $\bar{F}_1(x, \gamma) = 0$ where $\bar{F}_1$ is polynomial in x and affine in γ. Extracting the coefficients of $\bar{F}_1(x, \gamma) = 0$, we can build a matrix $\bar{F}_2(\gamma)$ which, together with $F_0(x^\star, \gamma) \succ 0$, can be solved with LMI solvers.

The Algorithm 1 shows our solution for the Implicit IDA-PBC with LMI solvers.

B. Selection of k

Case 1: Define $b_\perp^\top := \begin{bmatrix} S_\perp^\top & Z \end{bmatrix}^\top$ with Z any matrix complement to $S_\perp^\top$ for a full rank left annihilator $b_\perp$. Then, consider $k = \begin{bmatrix} I_{n_r - n_\lambda - n_u} & 0 \end{bmatrix}$. It is straightforward to see that the constraint matching equation (3c) is equivalent to $\mathcal{J}^\top b = S k_1$ for some $k_1(r) \in \mathbb{R}^{(n_\lambda + n_u)\times n_\lambda}$. In contrast, (8c) with this k is equivalent to $\mathcal{J}^\top b = S k_1$ and $\mathcal{J}^\top \mathcal{G} = S k_2$ for some $k_1(r) \in \mathbb{R}^{(n_\lambda + n_u)\times n_\lambda}$ and $k_2(r) \in \mathbb{R}^{(n_\lambda + n_u)\times n_u}$, which is more restrictive. See Section V-A.

[7] These models are not derived from a polynomial approximation.

Result: Implicit IDA-PBC Controller
Analyze the polynomial order of $\mathcal{M}$, $\mathcal{V}$ and S;
Select polynomial order for $\mathcal{M}_d$, $\mathcal{V}_d$ and $\mathcal{N}_i$;
Calculate $S_\perp$, $b_\perp$ and set k as Case 1;
Select $r^\star$;
Solve Matching conditions (8b), (8c) and (11) with (6);
if *(Solver converges)* ***and*** *($\mathcal{M}_d(r^\star)$ has full rank)* **then**
  Calculate u;
else
  Change order of $\mathcal{M}_d^{-1}$, $\mathcal{V}_d$, $\mathcal{N}_i$ or selection of k;
  Solve again conditions with the SDP solver;
end

Algorithm 1: Solution of Implicit IDA-PBC with LMIs.

Case 2: In some systems, the selection of k presented in Case 1 is not sufficient to satisfy the control objective, for instance the upright position of the cart-pole. Here, we propose to search for a more general structure of $k(r)$ that in the beginning only fulfills (8c). In this way, we multiply (9) on the right by $\begin{bmatrix} b & b_\perp^\top \end{bmatrix}$ resulting in (3c) and

$$\bar{k}^{-1}k = S_\perp \mathcal{J}^\top b_\perp^\top \left(b_\perp b_\perp^\top\right)^{-1}. \tag{10}$$

In the application, we first search symbolically for some $\tilde{\mathcal{M}}_d$ that meets (3c) and then calculate $\bar{k}^{-1}k$. Since multiplying (8) on the left by any square full rank matrix does no affect the results, we can simply use $\bar{k}^{-1}k$ instead of k.

C. Structure Selection for $\mathcal{W}_1$

Let us consider

$$\mathcal{W}_1 = \mathcal{M}_d\mathcal{M}^{-1}\sum_{i=1}^{n}\mathcal{N}_i\mathcal{M}^{-1}\rho e_i^\top\mathcal{M}_d$$

with $\mathcal{N}_i(r) = -\mathcal{N}_i^\top(r) \in \mathbb{R}^{n_r\times n_r}$, and $e_i \in \mathbb{R}^{n_\lambda}$ unitary column vectors, then we fulfill (5b) and rewrite (8a) as

$$\sum_{i=1}^{n} b_\perp\left(a_{ij1}+a_{ij2}+a_{ij3}\right)b_\perp^\top = 0, \tag{11}$$

$$a_{ij1} = v_j^\top k b_\perp \mathcal{M}\mathcal{M}_d^{-1}e_i\partial_{r_i}(\mathcal{M}),\ a_{ij2} = \mathcal{N}_i^\top b_\perp^\top k^\top v_j e_i^\top \mathcal{M},$$
$$a_{ij3} = v_j^\top k b_\perp e_i \mathcal{M}\partial_{r_i}(\mathcal{M}_d^{-1})\mathcal{M},$$

where $v_j \in \mathbb{R}^{n_r-n_\lambda-m}$ are unitary column vectors.[8] Furthermore, if $\mathcal{M}_d$ is constant and $S_\perp\partial_r^\top(\mathcal{M}^{-1}\rho) = 0$, then (11) can be satisfied for arbitrary skew symmetric matrices $N_i(r) \in \mathbb{R}^{(n_\lambda+m)\times(n_\lambda+m)}$ with

$$\mathcal{N}_i = \mathcal{M}\mathcal{M}_d^{-1}SN_iS^\top\mathcal{M}_d^{-1}\mathcal{M}. \tag{12}$$

V. SIMULATIONS

In the following examples we use the previous results to find stabilizing controllers for the portal crane and the cart-pole systems. The algorithm is processed in Matlab with SOSTOOLS and SDPT3. SOSTOOLS is a Matlab toolbox specialized on the SOS method [20]. Even though, here we do not directly use SOS, this toolbox provides a simple environment to work with polynomial matrices and equalities (obtaining $\bar{F}_2(\gamma)$). In addition, to guarantee strict inequalities in the Semidefinite Programming (SDP) solver, we add $10^{-5}I_{n_r-n_\lambda}$ in the right hand side of (6b) and (6a). For better visibility, values presented in this paper have been rounded to three decimals.

[8]Note that the kinetic matching equation (11) is independent of ρ.

A. Portal Crane

Let us consider a portal crane system with massless rope and fix distance l,[9] see Fig. 1. The implicit model without dissipation is given by (1), with $\Phi = \frac{1}{2}\left(x_p^2+y_p^2+z_p^2-l^2\right)$, $\mathcal{G} = \begin{bmatrix}0_{2\times3} & I_2\end{bmatrix}^\top$, gravity constant g_r,

$$\mathcal{M} = \begin{bmatrix} m_p & 0 & 0 & m_p & 0\\ 0 & m_p & 0 & 0 & m_p\\ 0 & 0 & m_p & 0 & 0\\ m_p & 0 & 0 & m_c+m_p & 0\\ 0 & m_p & 0 & 0 & m_c+m_p\end{bmatrix},\quad r = \begin{bmatrix}x_p\\ y_p\\ z_p\\ \tilde{x}_c\\ \tilde{y}_c\end{bmatrix},$$

and $\mathcal{V} = g_r m_p z_p$, where x_p, y_p and z_p are the Cartesian positions of pendulum, $\tilde{x}_c = x_c - x_c^\star$, $\tilde{y}_c = y_c - y_c^\star$, and x_c and y_c are the cart positions in a horizontal plane.

We shall determine an implicit IDA-PBC stabilizing controller. For this, we select

$$k = \begin{bmatrix}I_2\\ 0_2\end{bmatrix}^\top,\ b_\perp = \begin{bmatrix}S_\perp\\ Z^\top\end{bmatrix},\ S_\perp = \begin{bmatrix}-z_p & 0 & x_p & 0 & 0\\ 0 & -z_p & y_p & 0 & 0\end{bmatrix}$$

and solve the conditions of Prop. 4 in SOS with $m_p = m_c = l = 1$, $r^\star = \begin{bmatrix}0 & 0 & -l & 0 & 0\end{bmatrix}^\top$, a constant $\mathcal{M}_d$ and $\mathcal{V}_d$ a polynomial of maximum degree 2. Note that $S_\perp\partial_r^\top(\mathcal{M}^{-1}\rho) = 0$; therefore, we do not require to solve (11) and $\mathcal{N}_i$ is parameterized by (12). The results from SOS are: $\mu^\star = 1.17$,

$$\mathcal{M}_d^{-1} = \begin{bmatrix} 3.16 & 0 & 0 & -2.69 & 0\\ 0 & 3.16 & 0 & 0 & -2.69\\ 0 & 0 & 0.466 & 0 & 0\\ -2.69 & 0 & 0 & 2.69 & 0\\ 0 & -2.69 & 0 & 0 & 2.69\end{bmatrix}\ \text{and}$$

$$\mathcal{V}_d = 2.12(\tilde{x}_c^2+\tilde{y}_c^2)+1.7(x_p^2+y_p^2+z_p^2)+4.57z_p+2.87.$$

Figure 2 illustrates the behavior of the closed loop system with $K_u = \mathrm{diag}(2,2,0)$, $\mathcal{W}_1 = 0$, $x_p(0) = y_p(0) = x_c(0) = y_c(0) = \tilde{x}_c(0) = \tilde{y}_c(0) = 0$, $z_p(0) = -l$, $\rho(0) = 0$ and set points (represented by dotted lines) of $x_c^\star = 1$ m and $y_c^\star = 0.5$ m. We omit z_p in the illustration because it points

[9]If l is not fix, there is no constraint.

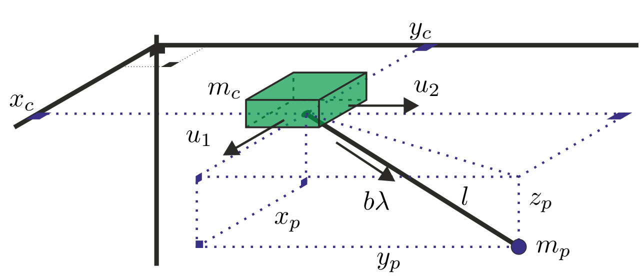

Fig. 1. Portal Crane system.

downwards and is fully determined by $\Phi(r) = 0$. Clearly, the states converge to $r^\star$ and the desired Hamiltonian decreases monotonically.

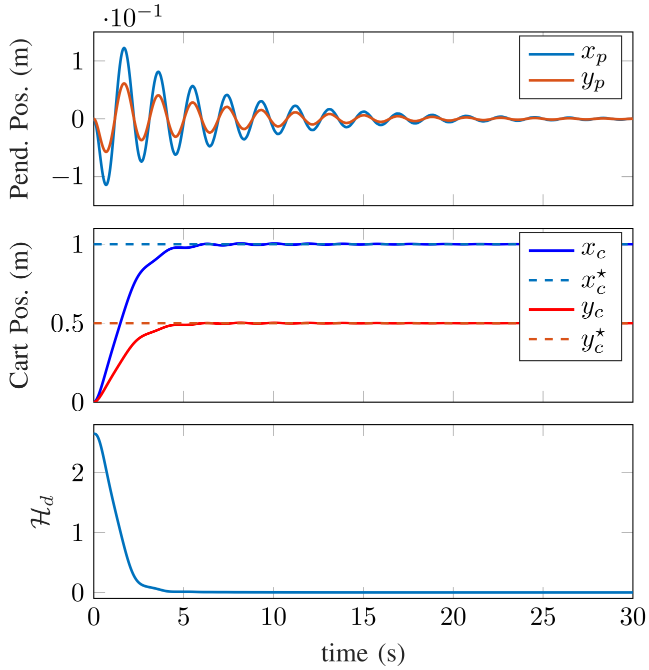

Fig. 2. Portal Crane: Response under $x_c^\star = 1$ m and $y_c^\star = 0.5$ m.

Finally, we also note that the solution obtained from the SDP solver satisfies $\mathcal{J}b = bQ$ for some square full rank matrix $Q(r)$. As a consequence, it is also possible to implement the output feedback law proposed in [10].

B. Cart-Pole

Now, we consider a cart-pole system with massless bar and without friction as shown in Fig. 3. The implicit model (1) has $\Phi = \frac{1}{2}\left(x_p^2 + y_p^2 - l^2\right)$, potential energy $\mathcal{V} = g_r m_p y_p$, gravity constant g_r, pendulum length l, $\tilde{x}_c = x_c - x_c^\star$,

$$r = \begin{bmatrix} x_p \\ y_p \\ \tilde{x}_c \end{bmatrix},\ \mathcal{M} = \begin{bmatrix} m_p & 0 & m_p \\ 0 & m_p & 0 \\ m_p & 0 & m_c + m_p \end{bmatrix}, \text{ and } \mathcal{G} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

1) Constant $\mathcal{M}_d^{-1}$: Following the same procedure as in the portal crane ($k = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $S_\perp = \begin{bmatrix} -y_p & x_p & 0 \end{bmatrix}$ and $\mathcal{V}_d$ a polynomial of maximum degree 2) for $r^\star = \begin{bmatrix} 0 & l & 0 \end{bmatrix}^\top$, we observe that the SDP solver in SOSTOOLS does not

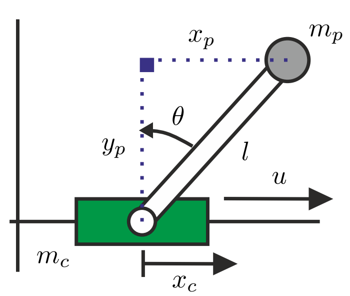

Fig. 3. Cart pole system.

converge, i.e., there is no solution for the upright position of the pendulum.[10] Therefore, we use the second case for the selection of k with $b_\perp = \begin{bmatrix} S_\perp^\top & \mathcal{G} \end{bmatrix}^\top$ and find $k = \begin{bmatrix} 1 & a_1 y_p \end{bmatrix}$, $a_1 \in \mathbb{R}$. Consider $m_p = m_c = l = 1$. Numerical experimentation then shows that the SDP solver converges if we select $a_1 > 1$. Picking $a_1 = 2$, we obtain

$$\mathcal{M}_d^{-1} = \begin{bmatrix} 3.06 & 0 & -0.954 \\ 0 & -0.195 & 0 \\ -0.954 & 0 & 0.318 \end{bmatrix}, \quad \mu^\star = 0.827,$$
$$\mathcal{V}_d = 0.425x_c^2 + 1.7x_c x_p + 2.24x_p^2 + 0.544y_p^2 - 1.92y_p + 1.37.$$

Eliminating constraints (2b) and forces $b_d(r)\lambda_d$ (reduction to explicit representation [10]), we can see that despite $\mathcal{M}_d$ being sign indefinite, the target mass matrix in explicit coordinates, given by M_d, is positive definite. Furthermore, if $\mathcal{V}_d$ has maximum degree 2, it is possible to find an equivalence with the PID-PBC [10].

2) Polynomial $\mathcal{M}_d^{-1}$: Now, we select $\mathcal{V}_d$, $\mathcal{W}_1$ and $\mathcal{M}_d^{-1}$ polynomial functions of third, first and second order, respectively, where $\mathcal{M}_d^{-1}$ has x_p and y_p as arguments. Solving the conditions of Prop. 4 in SOSTOOLS with (11) replaced by (8a), yields: $\mu^\star = 1.13$,

$$\mathcal{M}_d^{-1} = \begin{bmatrix} a_2 & a_5 & a_3 \\ a_5 & -0.012x_p^2 - 0.037y_p^2 - 0.246 & 0.024x_p y_p \\ a_3 & 0.024x_p y_p & a_4 \end{bmatrix},$$
$$\mathcal{N}_1 = \begin{bmatrix} 0 & 0 & 0.124x_p \\ 0 & 0 & -0.124y_p - 0.088 \\ -0.124x_p & 0.124y_p + 0.088 & 0 \end{bmatrix},$$
$$\mathcal{N}_2 = \begin{bmatrix} 0 & 0 & -0.244y_p - 0.153 \\ 0 & 0 & 0 \\ 0.244y_p + 0.153 & 0 & 0 \end{bmatrix},$$
$$\mathcal{N}_3 = \begin{bmatrix} 0 & 0 & -0.001x_p \\ 0 & 0 & 0.001y_p - 0.022 \\ 0.001x_p & 0.022 - 0.001y_p & 0 \end{bmatrix},$$
$$a_2 = 0.045x_p^2 + 2.35y_p^2 + 2.36y_p + 2.39,$$
$$a_3 = 0.005x_p^2 - 0.78y_p^2 - 0.787y_p - 0.714,$$
$$a_4 = -0.01x_p^2 + 0.26y_p^2 + 0.262y_p + 0.238,$$
$$a_5 = -0.049x_p y_p, \text{ and}$$
$$\mathcal{V}_d = 0.527x_c^2 + 2.11x_c x_p - 0.122x_p^2 y_p + 2.93x_p^2 - 0.122y_p^3 + 0.82y_p^2 - 2.41y_p + 1.71.$$

Figure 4 shows the behavior of the closed loop system under two control laws: u_1 calculated from Section V-B.1 with $K_u = \text{diag}(20, 0)$ and u_2 obtained from Section V-B.2 with $K_u = \text{diag}(4, 0)$. We define $x_p = l\sin\theta$, $y_p = l\sin\theta$ and consider initial conditions $\theta(0) = 30$deg, $x_c(0) = \tilde{x}_c(0) = 0$ m and $\rho(0) = 0$. Additionally, we include in 15s a set point (represented by a dotted line) of $x_c^\star = 2$m.

Since there is no optimization objective, we cannot yet draw from Fig. 4 any conclusion related to which control law

[10]Using $k = \begin{bmatrix} 1 & 0 \end{bmatrix}$ it is possible to find a solution for the downward position of the pendulum.

is better. However, selecting $\mathcal{M}_d$ polynomial allows more freedom in the adjustment of the algorithm, which under adequate additional inequalities may, e.g., increase robustness or the region of convergence. Finally, no equivalence with the PID-PBC has been found in this case.

VI. CONCLUSIONS

We simplify the conditions to implement (locally) the total energy shaping implicit IDA-PBC presented in [10]. Additionally, we introduce equivalent matching equations which take advantage of the typical polynomial structure of implicit port-Hamiltonian systems modeled in Euclidean space and, under some parametrization (using an additional matrix k), the new matching equations allow us to formulate the problem as LMIs with a user-defined polynomial order in the desired Hamiltonian. Thus, we extend the algorithm of [10], solving the kinetic matching equations for a possibly non-constant target inertia matrix, while still avoiding the need to solve PDEs.

The polynomial equalities and LMIs are solved with SOSTOOLS and SDPT3. Additionally, discussions for two parametrizations of the equivalent matching equations are also presented. The approach is shown on two standard underactuated mechanical examples: the cart-pole and portal crane system.

Since no performance or optimization objective for the SDP solver has been proposed yet, our future works focus on local optimality and increasing the region of convergece.

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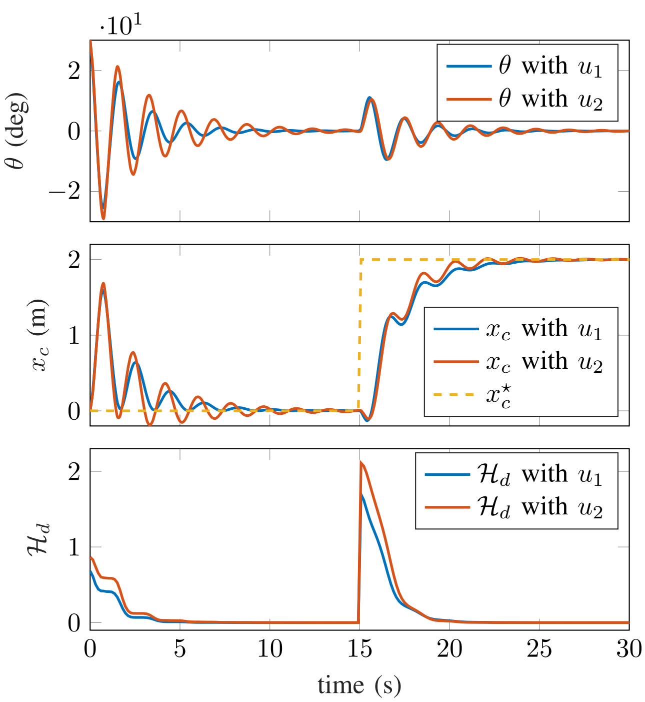

Fig. 4. Cart-Pole: Response comparison for both controllers under $\theta(0) = 30$ deg. and $x_c(0) = 0$.

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