Robust semiglobal stabilization of the second order system by relay feedback with an uncertain variable time delay

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***Abstract*— We present sufficient conditions for robust relay delayed *semiglobal* stabilization of second order systems, which relate the upper bound to an uncertain time delay and the parameters of the plant. We also suggest an algorithm of delayed relay control gain adaptation for *semiglobal* stabilization, which is based on *delayed information about the sign of the controlled variable only*. The proposed algorithm suppresses *bounded uncertainties in the time delay*, that is, being designed for the upper bound of uncertainty in the time delay, the control law ensures semiglobal stabilization independently of any variable time delay obeying the given upper bound.**

I. Introduction

We present an abbreviated version of the paper [18].

A. Statement of the problem

We study the control problem for the second order system

$$\alpha\ddot{x}(t) = -\beta\dot{x}(t) + F(x(t), t) + u \tag{1}$$

with positive constants α and β and some function $F(x, t)$, satisfying

$$F \in C^1(R^2), \quad \sup\left|\frac{\partial F}{\partial x}\right| < \infty\ . \tag{2}$$

The uncontrolled system

$$\alpha\ddot{x} = -\beta\dot{x} + F(x, t)$$

may be unstable, as, for example, in the case $F(x, t) = kx$, $k > 0$, and we propose to stabilize it by a negative feedback of relay type:

$$u = -K(t) \cdot \operatorname{sign} x(t - \tau), \tag{3}$$

with a controllable bounded magnitude $K(t) > 0$ and a positive variable uncertain delay τ, assumed to be a measurable function of t obeying the condition

$$0 < \tau_0(t) \le \tau(t) \le h = \text{const}, \quad t \ge 0, \tag{4}$$

where $\tau_0(t)$ is a positive non-increasing function.

Our aim is to design a piece-wise constant controller $K(t)$, which provides a robust semiglobal stabilization of the oscillation magnitude of the solutions to system (1), (3).

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B. Motivation

For the motivation, we point out that time delay in control systems is usually present and must be taken into account. In practice, many systems with time delay naturally admit relay controllers, in particular,

- *systems which can work in switching modes*, for example power converters (see, for example, [17]);
- *systems with measuring devices that work in the switching mode and have time delay*, for example, controllers of exhausted gas in the fuel injector automotive control systems [13], which act with delay and, moreover, generate relay signals only;
- *sliding mode systems with delayed actuators*, for example, the stabilizers of the fingers for an underwater manipulator ([2]);
- *mathematical biology systems* as, for example, those considered in [10], [11].

It has been shown in [4] that, in the simplest one-dimensional relay control systems with a constant delay, only oscillatory solutions can occur. Moreover, any such solution becomes periodic after a finite time interval, but only slowly oscillating solutions are stable. The latter property is used to design an algorithm controlling the motion amplitudes.

P.I. control algorithms for the amplitude control in one-dimensional relay systems with delay in the input have been suggested in [1]. A Pade approximation of delay that reduces the relay delay output tracking problem to the sliding mode control for non-minimum phase system was suggested in [14]. Delayed relay control algorithms, suggested in [5], [6], allow one to reach local and nonlocal stabilization of oscillations amplitudes for MIMO systems, respectively, with the use of the delayed value of the magnitude of a current trajectory.

In [9], periodic properties of second order systems via relay delayed controllers based on the sub-optimal control algorithm were investigated, whereas the article [3] studies oscillations in first order systems, containing external forcing in the relay delayed control element.

C. The Main Result

Restrictions to the non-linear element. Throughout the paper we impose the following bound of the nonlinear term $F(x, t)$ of equation (1):

$$0 \le \frac{F(x, t) - F(0, t)}{x} \le k_0, \quad x \ne 0, \quad t \ge 0\ , \tag{5}$$

with some positive constant k_0. Furthermore, we separate between the two situations:

$$F(0,t) \equiv 0 \ , \tag{6}$$

and

$$|F(0,t)| \le \delta, \quad t \ge 0, \quad \delta = \mathrm{const} \in (0,1) \ , \tag{7}$$

in which the suggested controller and the respective solutions to (1), (3) behave differently.

The initial value problem and the definition of the discontinuous element. For system (1), (3), we state the initial value problem

$$x\big|_{[-h,0]} = \varphi, \ \dot{x}(0) = \dot{\varphi}(0), \quad \varphi \in C_0[-h,0], \tag{8}$$

by defining the initial data range to be the space $C_0[-h,0]$ of continuous functions $\varphi : [-h,0] \to \mathbf{R}$, differentiable at the origin. We equip $C_0[-h,0]$ with the norm

$$\|\varphi\| = \max_{[-h,0]} |\varphi(t)| + |\dot{\varphi}(0)| \ . \tag{9}$$

Realistic relay controllers take the only values ± 1, and, taking into account possible vanishing of $\varphi \in C_0[-h,0]$ along intervals, we admit

$$\operatorname{sign} x(t) = \ \zeta(t), \quad \text{as} \quad x(t) = 0, \tag{10}$$

where $\zeta(t)$ is any measurable function with $|\zeta(t)| = 1$ and consider the solutions to system (1), (3) in the sense of Carathéodory (see, for example, [7]). [1]

Then we have

Lemma 1: The equation

$$\alpha \ddot{x}(t) = -\beta \dot{x}(t) + F(x(t),t) - \operatorname{sign} \ x(t-\tau) \ ,$$

satisfying (2), (4), and (7), with initial condition (8) supplied with (10), has a unique continuous solution $x_\varphi(t)$, $t \in [-h,\infty)$. Moreover, x_φ is differentiable in the interval $(0,\infty)$, its derivative is absolutely continuous and differentiable almost everywhere.

We only notice that the lower bound to τ in (4) is needed for an accurate justification of the existence and uniqueness of the solution x_φ.

Remark 1: The solutions $x(t)$ to system (1), (3), considered in the sequel will satisfy the condition $|F(x(t),t)| < K(t)$, and thus, in the same way as in Lemma 1, the zero locus of such a solution $x(t)$ in the interval $t \ge 0$ will have zero measure whatever the zero locus of the initial function $\varphi(t) \in C_0[-h,0]$ is, and hence the results do not depend on the choice of the function $\zeta(t)$ in (10) for $t \ge 0$. In particular, shifting the initial interval to $[0,h]$, one obtains the zero locus of zero measure for the (new) initial function, getting rid of any dependence of the function $\zeta(t)$. In addition, $\dot{x}(t)$ turns to be differentiable almost everywhere.

The statement. We give here a general formulation, solving the stated problem, then present ideas leading to the result, and finally, in section III, provide precise expressions for all the parameters in the assertion.

[1]The sliding modes can not occur in the considered class of the systems. That is why it is not necessary to use more complicated definitions of the solutions for relay systems with delay (cf. [8],[12]).

Main Result. *Given system (1), (3) with $F(x,t)$ satisfying (5), under certain restrictions to α, β, h, δ, and k_0, there exist positive constants c, T_0, m, and $\rho < 1$ such that*

(1) in the case (6), for

$$K(t) = \rho^n, \quad nT_0 \le t < (n+1)T_0, \quad n = 0,1,2,\dots \ , \tag{11}$$

all the solutions with $\max\{|x(0)|, |\dot{x}(0)|\} < c$ *exponentially decay to zero;*

(2) in the case (7), for

$$\begin{aligned} &K(t) = \rho^n, \quad nT_0 \le t < (n+1)T_0, \ 0 \le n < m, \\ &K(t) = \rho^m, \quad t \ge mT_0, \end{aligned} \tag{12}$$

all the solutions with $\max\{|x(0)|, |\dot{x}(0)|\} < c$ *come to a neighborhood of zero, whose size is proportional to δ.*

In section III-A, we provide explicit formulas for all the parameters α, β, h, δ, and k_0, and in section V, we make a numerical simulation.

The meaning of Main Result is that, whenever the parameters of the system (1) and the controller delay τ satisfy some explicitly written restriction, a control presented by a step function $K(t)$ with *a priory* fixed switch moments and amplitudes brings solutions to a prescribed neighborhood of zero. In other words, we propose an algorithm for a robust *semiglobal* stabilization of the oscillation magnitude, based on a retarded relay switching of the control gain, which requires only the knowledge of the sign for the controlled variable in the past and allows us to reject uncertainty in the time delay.

D. The ideas behind Main Result

The idea of a piece-wise constant control function $u(t)$ can be traced back to [4], where such a controller, acting with a constant delay, has been used for an exponential stabilization of oscillations in the first order system

$$\dot{x}(t) = F(x(t),t) - \operatorname{sign} x(t-h)$$

with F satisfying (5) and (6). The key observation was that, if $k_0 h < \log 2$, then the solutions starting in a small neighborhood of zero cannot reach some critical value $|x| = M_0$ during the time interval h, and then must return to the zero level, that is, remain bounded and oscillating. Furthermore, for such solutions, $\sup |x(t)| < M_1 < M_0$, and hence, switching the magnitude of sign from 1 to $\rho = M_1/M_0 < 1$ at the moment t^* with $x(t^*) = 0$ and making change $x = \rho x^{(1)}$, we come to an equation

$$\dot{x}^{(1)}(t) = F^{(1)}(x^{(1)}(t),t) - \operatorname{sign} x^{(1)}(t-h)$$

with $F^{(1)}$ again satisfying (5) and (6), which in turn means $|x(t)| < \rho M_0$ as $t \ge t^*$. Performing inductively the same procedure, one obtains exponentially decreasing solutions. However, that controller was depending on the term $F(x,t)$ and on the current solution, which made it hard to realize in practice. This difficulty has been resolved in [16], where a similar piece-wise constant controller acting with a variable uncertain bounded delay and having a priori fixed switches provided an exponential decay of solutions with a sufficiently small initial values.

In a similar way we obtain Main Result for the second order system (1), (3). The background property, established in [15], states that, under certain restrictions on the positive parameters $\alpha, \beta, k, K, h, c$, the solutions to the equation

$$\alpha\ddot{x} = -\beta\dot{x} + kx - K \cdot \operatorname{sign} x(t-h) \ ,$$

which obey the initial conditions $x(0) = 0$, $|\dot{x}(0)| < c$, remain bounded by a constant M, proportional to K, and, moreover, the derivatives $\dot{x}(t^*)$ for all $t^* > 0$, $x(t^*) = 0$ belong to a *smaller* range $(-c_1, c_1)$, where $c_1 < c$. Here we extend this fact to the case of arbitrary functions $F(x,t)$ with bounded values and derivative, and a variable uncertain delay $\tau(t)$. So, again after a suitable period of time, we switch the controller magnitude from K to ρK with some $\rho \in (c_1/c_0, 1)$, and make change $x = \rho x^{(1)}$, coming to an equation for $x^{(1)}$, analogous to (1), (3) and satisfying the hypotheses, which provide $|x^{(1)}| < M$ and $|\dot{x}^{(1)}| < c$, and, in particular, $|x(t)| < \rho M$ for large $t > 0$.

To find suitable bounds to the given data, we model the "worst" behavior of a solution to (1), (3), which means that the absolute value $|x|$ maximally grows against the negative feedback u intended to bring the solution to the zero level. That is, if a solution starts at zero with some, say, positive derivative, we assume that $F(x,t) = \delta + kx$ and $\tau = h$, so that the feedback u remains positive on the largest possible interval of length h. Then we assume that the value of the control undergoes a switch only after a period of time h has elapsed, and we know that reversing the sign of the control will eventually force the solution to reach a maximum. When the maximum value is attained, we take $F(x,t) = -\delta$ and wait until the solution reaches the next zero. We call the pieces of that "worst" solution *majorating functions*. They are treated in the next section in order to precisely state the sufficient conditions for the existence of the controller proposed in Main Result, and these conditions finally reduce to the claim that the absolute value of the derivative of the "worst" solution at its zero is *strictly greater* than that value at the next zero.

Acknowledgements. L. Fridman gratefully acknowledges the financial support of this work by the Mexican CONACyT (Consejo Nacional de Ciencia y Tecnologia), grant no. 56819, and the Programa de Apoyo a Proyectos de Investigacion e Innovacion Tecnologica (PAPIIT) UNAM, grant no. IN111208. Programa de Apoyo a Proyectos Institucionales para el Mejoramiento de la Ensenanza (PAPIME), UNAM, grant PE100907.

II. MAJORATING FUNCTIONS

A. Definition of the majorating functions

First, we point out that restriction (5) on the nonlinearity $F(x,t)$ comes from the comparison of system (1), (3) with the equation $\alpha\ddot{x} = -\beta\dot{x} + k_0 x - \operatorname{sign} x(t-\tau)$, and this makes natural to introduce the roots $\lambda_1 > 0 > \lambda_2$ of the characteristic equation $\alpha\lambda^2 + \beta\lambda - k_0 = 0$, which will play an important role in the further consideration.

In order to deal with uncertainty, we introduce a majorating function for the actual response $x(t)$. This function is intended to model "the worst type of behavior" of the stable solutions to (1), (3). As it was previously stated, such solutions are periodic and slow. Assuming that the distance between the neighboring zeroes of $x(t)$ is greater than h, we can divide the interval between such zeros into three parts:

1) an interval between the current zero and the (first) control switch,
2) an interval between the control switch and the global extremum,
3) the remaining part from the extremum to the next zero.

For the first interval consider the equation

$$\alpha\ddot{y}(t) = -\beta\dot{y}(t) + k_0 y(t) + 1 + \delta.$$

We assume that the control switch is delayed by h from the current zero, and since we are considering the upper lobe, the initial derivative is positive. Hence we impose

$$0 \le t \le h, \quad y(0) = 0, \ \dot{y}(0) = a,$$

where a is a non-negative parameter.

The family of solutions is

$$y_{\delta,a}(t) = P_{\delta,a}(\lambda_1, \lambda_2) + P_{\delta,a}(\lambda_2, \lambda_1) - (1+\delta)/k_0 \ ,$$

$$P_{\delta,a}(\lambda, \mu) = \frac{ak_0 - \mu(1+\delta)}{k_0(\lambda - \mu)} \, e^{\lambda t} \ .$$

For the second interval we introduce the solution

$$z_{\delta,a}(t) = Q_{\delta,a}(\lambda_1, \lambda_2) + Q_{\delta,a}(\lambda_2, \lambda_1) + (1-\delta)/k_0 \ , \quad (13)$$

$$Q_{\delta,a}(\lambda, \mu) = \frac{2e^{-\lambda h} - 1 - \delta - \alpha a \lambda}{\alpha\lambda(\mu - \lambda)} \, e^{\lambda t} \ ,$$

of the equation

$$\alpha\ddot{z}(t) = -\beta\dot{z}(t) + k_0 z(t) - 1 + \delta, \quad z(h) = y_{\delta,a}(h), \ \dot{z}(h) = \dot{y}_{\delta,a}(h).$$

Suppose that $2e^{-\lambda_1 h} - 1 > 0$,

$$\delta < 2e^{-\lambda_1 h} - 1 \quad (14)$$

and

$$a < (2e^{-\lambda_1 h} - 1 - \delta)/(\alpha\lambda_1) \ . \quad (15)$$

This means, in particular, that the coefficients of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ in (13) are negative. Hence $z_{\delta,a}(t)$ is a concave function, which in view of $\dot{z}_{\delta,a}(h) = \dot{y}_{\delta,a}(h) > 0$ has a unique maximum in (h, ∞). The maximum occurs at the time moment

$$t_{\delta,a} = \frac{1}{\lambda_1 - \lambda_2} \log \frac{2e^{-\lambda_2 h} - 1 - \delta - \alpha a \lambda_2}{2e^{-\lambda_1 h} - 1 - \delta - \alpha a \lambda_1} \ , \quad (16)$$

so, for $z_{\delta,a}(t)$ we add the restriction

$$h \le t \le t_{\delta,a}.$$

To keep the notation simple, we will set

$$\sigma_\delta = z_{\delta,a}(t_{\delta,a}) \ . \quad (17)$$

For the last interval we introduce the equation

$$\alpha\ddot{w}(t) = -\beta\dot{w}(t) - 1 - \delta.$$

In what follows, we will be more concerned about the value of the global extremum σ_δ, rather than the time of its occurrence $t_{\delta,a}$, so we add

$$w(0) = \sigma_\delta, \quad \dot{w}(0) = 0,$$

which defines a majorating function which is shifted in time. The solution is given by

$$w_{\delta,\sigma_\delta}(t) = \frac{\alpha(1+\delta)}{\beta^2}(1 - e^{-t\beta/\alpha}) - \frac{1+\delta}{\beta}\, t + \sigma_\delta, \quad (18)$$

which has a unique positive root t'.

Now we can build the majorating function. For any perturbation with a bound δ satisfying (14) and an initial derivative a satisfying (15), the function

$$\phi_{\delta,a}(t) = \begin{cases} y_{\delta,a}(t), & 0 \le t \le h \\ z_{\delta,a}(t), & h \le t \le t_{\delta,a} \\ w_{\delta,\sigma_\delta}(t - t_{\delta,a}), & t_{\delta,a} \le t \le t' + t_{\delta,a} \end{cases} \quad (19)$$

bounds from above the solutions to (1), (3) and their derivatives. We shall call $\phi_{\delta,a}(t)$, *the worst solution* to (1), (3).

B. Properties of the majorating functions

The next property will play a key role in the argumentation that follows.

Definition 1: A function $\phi_a(t) \in C_1$ is said to be *differentially contractive* (DC) if whenever it starts at a zero with a derivative belonging to some interval, its derivative at the next zero belongs to a smaller interval.

Notice that $\phi_{\delta,a}$ is continuous. Its initial derivative a is taken from the interval (15), and we want its terminal derivative

$$\xi_\delta(\sigma_\delta) = |\dot{w}_{\delta,\sigma_\delta}(t')|$$

to belong to a smaller interval.

To fulfill the DC property, we first use (18) to estimate t'

$$\frac{\alpha(1+\delta)}{\beta^2}\,(1 - e^{-t'\beta/\alpha}) = \frac{1+\delta}{\beta}\, t' - \sigma_\delta.$$

Due to $\dot{z}_{\delta,a}(t_{\delta,a}) = 0$ and $\ddot{z}_{\delta,a}(t_{\delta,a}) < 0$, we have

$$\sigma_\delta = \frac{1 - \delta + \beta\dot{z}_{\delta,a}(t_{\delta,a}) + \alpha\ddot{z}_{\delta,a}(t_{\delta,a})}{k_0} < \frac{1-\delta}{k_0}. \quad (20)$$

Hence $t' \le \theta$, where θ is the positive root of the equation

$$\frac{\alpha(1+\delta)}{\beta^2}\,(1 - e^{-\theta\beta/\alpha}) = \frac{1+\delta}{\beta}\,\theta - \frac{1-\delta}{k_0}. \quad (21)$$

Notice that, given $\alpha, \beta, \delta, k_0$, equation (21) always has a unique positive root θ, since the left-hand side is a positive concave function of θ, and the right-hand side is an increasing linear function of θ, negative at the origin. Next, we have that

$$\xi_\delta(\sigma_\delta) = \frac{1+\delta}{\beta}\,(1 - e^{-t'\beta/\alpha}) \le \frac{1+\delta}{\beta}\,(1 - e^{-\theta\beta/\alpha}). \quad (22)$$

In view of the last inequality and (15) it is easy to see that $\phi_{\delta,a}$ fulfills DC if

$$\frac{1+\delta}{\beta}\,(1 - e^{-\theta\beta/\alpha}) < \frac{2e^{-\lambda_1 h} - 1 - \delta}{\alpha\lambda_1}. \quad (23)$$

To understand inequality (23), consider the equality

$$\frac{1+\delta}{\beta}\,(1 - e^{-\theta(k)\beta/\alpha}) = \frac{2e^{-\lambda_1(k)h} - 1 - \delta}{\alpha\lambda_1(k)} \quad (24)$$

as an equation to the unknown k with fixed α, β, δ. Here the left-hand side is a bounded positive function of k, whereas the right-hand side drops from infinity to negative values as k grows from zero to infinity. Hence (24) has positive roots, and the minimal one among them we denote by $k_{\min}$. So, finally, we reduce (23) to

$$k_0 < k_{\min}, \quad (25)$$

which guaranties the DC property.

Remark 2: According to (20) the extremum σ_δ is bounded from above. We shall call that bound $\sigma_{\max}$, i.e.

$$\sigma_\delta < \sigma_{\max} = (1-\delta)/k_0 \ .$$

Suppose that the extremum attains the maximum value in the current period, in view of (20), the extremum at the following period satisfies

$$\sigma_\delta^{(1)} = z_{\delta,\xi_\delta(\sigma_{\max})}(t_{\xi_\delta(\sigma_{\max})}) < \sigma_{\max},$$

III. Main Results in detail

Definition 2: Denote by $\Phi_{\delta,a}$ the set of functions $\varphi \in C_0[-h, 0]$ such that either

$$\varphi^{-1}(0) \ne \emptyset, \quad |\varphi(0)| \le y_{\delta,a}(-t^*), \quad |\dot{\varphi}(0)| \le \dot{y}_{\delta,a}(-t^*),$$

where $t^* = \max\varphi^{-1}(0) > -h$ or

$$\varphi\big|_{(-h,0]} \ne 0, \quad |\varphi(0)| \le z_{\delta,a}(t^*), \quad |\dot{\varphi}(0)| \le \dot{z}_{\delta,a}(t^*)$$

for some $t^* \in [h, t_{\delta,a}]$.

A. Perturbations that vanish at the origin

Assume that $\delta = 0$. In order to simplify notations, in this case we always skip the subindex δ (i.e., 0) in the notations for $t, \xi, \sigma, \Phi, x, y, z$.

Introduce the following parameter. Given

$$0 < a < b < \frac{2e^{-\lambda_1 h} - 1}{\alpha\lambda_1}\ ,$$

set

$$\rho(a,b) = \frac{\alpha a(\lambda_1 e^{\lambda_1 h} - \lambda_2 e^{\lambda_2 h}) + (e^{\lambda_1 h} - e^{\lambda_2 h})}{\alpha b(\lambda_1 e^{\lambda_1 h} - \lambda_2 e^{\lambda_2 h}) + (e^{\lambda_1 h} - e^{\lambda_2 h})}. \quad (26)$$

Clearly, $\rho(a,b) < 1$. Then introduce

$$\rho = \rho(\xi(\sigma^{(1)}), \xi(\sigma_{\max}))\ . \quad (27)$$

Notice that ρ is defined properly, since ξ is a strictly increasing function.

Theorem 1: Assume that $F(x,t)$ and $\tau(t)$ satisfy (2), (4), (5), (6), and (25) with $\delta = 0$. Let a constant c satisfy

$$0 < c < \frac{2e^{-\lambda_1 h} - 1}{\alpha\lambda_1}. \qquad (28)$$

Put

$$K(t) = \begin{cases} 1, & if \quad 0 \le t < t_c, \\ \rho^n, & if \quad nt_{\xi(\sigma_{\max})} \le t - t_c < (n+1)t_{\xi(\sigma_{\max})}, \\ & \quad n = 0, 1, 2, \ldots \end{cases}$$

where ρ is defined by (27), t_c and $t_{\xi(\sigma_{\max})}$ are the roots of $z_c(t)$ and $z_{\xi(\sigma_{\max})}(t)$ respectively.

Then any solution $x_\varphi(t)$ to (1), (3), (8) with $\varphi \in \Phi_c$ obeys the restriction

$$|x_\varphi(t)| \le \frac{1}{k_0}\ \exp\left(-\left(\log\frac{1}{\rho}\right)\frac{t - t_c - t_{\xi_\delta(\sigma_{\max})}}{t_{\xi_\delta(\sigma_{\max})}}\right), \qquad (29)$$

$$t \ge t_c + t_{\xi_\delta(\sigma_{\max})}.$$

B. Perturbations that do not vanish at the origin

In realistic models, $F(0,t)$ does not vanish identically, so we'll consider the case $\delta \neq 0$, but we'll maintain restriction (14). In this case, one can drive the system in a finite time to a neighborhood of zero, proportional to δ. More precisely, we design a set of controllers, which depend on one continuous and one discrete parameter. The parameters can be chosen in their range according to the initial magnitude, the required rate of convergence and the size of the target neighborhood of zero. We only remark that one cannot optimize the two latter values simultaneously.

Given δ satisfying (14), the range of a positive parameter ε is defined by the inequality

$$\frac{1+\delta}{\beta}\,(1 - e^{-\theta\beta/\alpha}) + \frac{\delta}{\alpha\lambda_1} < \frac{2e^{-\lambda_1 h} - 1 - \varepsilon}{\alpha\lambda_1} \qquad (30)$$

Observe that (30) defines a non-empty interval, since it turns into (23) for $\varepsilon = 0$. Next we choose any natural $m \ge 1$ and put $q = q(\varepsilon, m)$ to be the positive root of the equation

$$\frac{1}{q}\left(\frac{1 - e^{-\theta\beta/\alpha}}{\beta} + \frac{(1-q)(e^{\lambda_1 h} - e^{\lambda_2 h})}{\alpha(\lambda_1 e^{\lambda_1 h} - \lambda_2 e^{\lambda_2 h})}\right)$$

$$+\frac{\delta}{q^m}\left(\frac{1 - e^{-\theta\beta/\alpha}}{\beta} + \frac{1}{\alpha\lambda_1}\right) = \frac{2e^{-\lambda_1 h} - 1 - \varepsilon}{\alpha\lambda_1}. \qquad (31)$$

Such a root does exist; furthermore, it is unique and belongs to the interval $(0,1)$. Indeed, the left-hand side of (31) monotonically decreases from infinity to the left-hand side of (30), whereas the right-hand sides of (30) and (31) coincide. Furthermore,

$$\frac{1}{q}\left(\frac{1 - e^{-\theta\beta/\alpha}}{\beta} + \frac{(1-q)(e^{\lambda_1 h} - e^{\lambda_2 h})}{\alpha(\lambda_1 e^{\lambda_1 h} - \lambda_2 e^{\lambda_2 h})}\right)$$

$$+\frac{\delta}{q^{m'}}\left(\frac{1 - e^{-\theta\beta/\alpha}}{\beta} + \frac{1}{\alpha\lambda_1}\right) \le \frac{2e^{-\lambda_1 h} - 1 - \varepsilon}{\alpha\lambda_1} \qquad (32)$$

for all $m' \le m$. At last, put

$$T(\varepsilon) = \frac{1}{\lambda_1}\ \log\frac{(1-\delta)(\lambda_1 - \lambda_2)}{-\lambda_2\varepsilon}. \qquad (33)$$

Theorem 2: Under the hypotheses (2), (4), (5), (7), and (25) with $\delta > 0$ satisfying (14), let ε obey (30). Put

$$K(t) = \begin{cases} q^s, & sT(\varepsilon) \le t < (s+1)T(\varepsilon),\ 0 \le s < m, \\ q^m, & t \ge mT(\varepsilon). \end{cases} \qquad (34)$$

Then any solution $x_\varphi(t)$ to (1), (3), (8) with $\varphi \in \Phi_{\delta,c}$, where

$$c = (2e^{-\lambda_1 h} - 1 - \delta - \varepsilon)/(\alpha\lambda_1)\ , \qquad (35)$$

obeys the restriction

$$|x_\varphi(t)| \le (q^m - \delta)/k_0 \quad \text{as} \quad t \ge mT(\varepsilon). \qquad (36)$$

IV. Control Algorithm

We shortly describe how to apply Theorems 1 and 2. One begins with a few common initial steps:

1) Given system (1), (3), obeying (2), (4), and (7) with known $h > 0$ and $\delta \ge 0$, we start by solving simultaneously the equations

$$\frac{\alpha(1+\delta)}{\beta^2}\,(1 - e^{-\theta_m\beta/\alpha}) = \frac{1+\delta}{\beta}\,\theta_m - \frac{1-\delta}{k_m}$$

$$\frac{1+\delta}{\beta}\,(1 - e^{-\theta_m\beta/\alpha}) = \frac{2e^{-\lambda_1(k_m)h} - 1 - \delta}{\alpha\lambda_1(k_m)}$$

with respect to positive unknowns k_m and θ_m.

2) Take the solution (k_m, θ_m) and verify that the given function $F(x,t)$ satisfies (5) with certain positive $k_0 < k_m$; then find the positive root θ of equation (21).
3) Compute the roots $\lambda_1 > 0 > \lambda_2$ of the characteristic equation, and check the validity of (14).

A. Perturbations that vanish at the origin

Perform steps (1) to (3) as described above and then do the following.

4) Pick a constant c, satisfying (28) and compute the values of $\xi(\sigma_{\max})$, t_c, $t_{\xi(\sigma_{\max})}$ and ρ using the formulas of Theorem 1. Verify that the initial function φ belongs to Φ_c as described in Definition 2.

Remark 3: It is possible to set a limit n^* to the maximum number of allowed switches of the controller, or to the time interval $t \le t^*$, when switches are allowed. Pick $n \le n^*$ or $n \le (t^* - t_c)/t_{\xi(\sigma_{\max})}$, respectively. The solution becomes bounded by $|x(t)| \le \rho^n/k_0$ after $t \ge t^*$.

B. Perturbations that do not vanish at the origin

Again perform steps (1) to (3) as above, and then proceed in the following way.

4) Pick a positive ε satisfying (30) and compute $T(\varepsilon)$ by (33).
5) For the last step there are three possibilities:
   a) Choose an upper bound m^* to the number of allowed switches of the controller, pick $m \le m^*$.
   b) Set the the size t^* of the time interval when switches are allowed, pick $m \le t^*/T(\varepsilon)$.

   In both cases solve equation (31) with respect to q. The solution will be bounded according to (36).

c) In this case we bring the solution to the $\delta(K_0+\kappa)$-neighborhood of zero, where

$$K_0\delta = \frac{2(\alpha\lambda_1(1-e^{-\theta\beta/\alpha})+\beta(1-e^{-\lambda_1 h}))}{k_0((2e^{-\lambda_1 h}-1)\beta-\alpha\lambda_1(1-e^{-\theta\beta/\alpha}))}\,\delta,$$

and κ is a (relatively) small prescribed positive parameter. Using (31) we compute

$$q = \frac{B_1+B_2}{C+B_2-(1-\varepsilon)/(k_0(K_0+\kappa)+1)}\ ,$$

where

$$B_1 = \frac{1-e^{-\theta\beta/\alpha}}{\beta}, \quad B_2 = \frac{(e^{\lambda_1 h}-e^{\lambda_2 h})}{\alpha(\lambda_1 e^{\lambda_1 h}-\lambda_2 e^{\lambda_2 h})},$$

$$C = (2e^{-\lambda_1 h}-1-\varepsilon)/(\alpha\lambda_1),$$

and, finally, put

$$m = \left[\frac{\log(\delta(k_0(K_0+\kappa)+1)/(1-\varepsilon))}{\log q}\right]+1.$$

V. Numerical example: Stabilization of inverted pendulum

Consider the stabilization problem of an inverted pendulum via a controller with uncertain delay. The oscillations of an inverted pendulum with unit mass with such a controller are described by equation

$$\ddot{x}+k\dot{x}-p\sin x+\delta = u(t-\tau(t)), \qquad (37)$$

where $k>0$ is a friction coefficient, $p=g/l>0$, δ uncertainty, τ is an uncertain time delay $0<\tau_0(t)\le\tau(t)\le h$. Consider the case, when $k=1$, $p=g/l=1.4$. In this case the equation (37) takes the form:

$$\ddot{x}(t) = -\dot{x}(t)+1.4\sin(x)+\delta-K(t)\mathrm{sign}\ (x(t-\tau(t))),$$

with $\tau=0.05+0.04\sin(t)$. It is clear that

$$\alpha=\beta=1, \quad \text{and} \quad F(x,t)=1.4\sin(x)+\delta,$$

and that the bound

$$0<\tau_0(t)\le\tau(t)\le h=0.1$$

holds.

We apply Theorem 2 for $\delta = 0.05$. The following parameters where obtained along the above algorithm:

$$\begin{array}{lll} k_m=2.2854 & k_0=1.5 & \lambda_1=0.8229 \\ \theta_m=1.0438 & \theta=1.3418 & \lambda_2=-1.8229 \\ & \delta<2e^{-\lambda_1 h}-1=0.7930 & \end{array}$$

Now we pick an $\varepsilon=0.15$ satisfying (30) and evaluate $T(\varepsilon)=2.696$. For the last step we choose $m=m^*=40$ and obtain $q=0.9975$.

VI. Conclusions

The dynamics of the second order systems with a delayed relay control is analyzed. Sufficient conditions for robust delayed relay *semiglobal* stabilization of second order systems are found. Such conditions relate to the upper bound of an uncertainty in time delay and the parameters of the plant. An algorithm for a delayed relay control with gain adaptation is suggested. The algorithm *is based on delayed information about the* ***sign*** *of controlled variable only*. The proposed algorithm suppresses *bounded uncertainties in the time delay:* once being designed for the upper bound of time delay in the given system, this control law ensures semiglobal stabilization for any constant or variable time delay within the given constraint.

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