Passivity of Nonlinear Incremental Systems: Application to PI Stabilization of Nonlinear RLC Circuits

Bayu Jayawardhana, Romeo Ortega, Eloísa García–Canseco and Fernando Castaños

***Abstract*— It is well known that if the linear time invariant system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\ \mathbf{y} = \mathbf{C}\mathbf{x}$ is passive the associated incremental system $\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\tilde{\mathbf{u}},\ \tilde{\mathbf{y}} = \mathbf{C}\tilde{\mathbf{x}}$, with $\tilde{(\cdot)} = (\cdot) - (\cdot)^\star$, $\mathbf{u}^\star, \mathbf{y}^\star$ the constant input and output associated to an equilibrium state $\mathbf{x}^\star$, is also passive. In this paper, we identify a class of nonlinear passive systems of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}\mathbf{u},\ \mathbf{y} = \mathbf{h}(\mathbf{x})$ whose incremental model is also passive. Using this result we then prove that general nonlinear RLC circuits with convex and proper electric and magnetic energy functions and passive resistors with monotonic characteristic functions are globally stabilizable with linear PI control.**

I. PROBLEM FORMULATION

In many control applications one is interested in operating the system around a *non–zero* equilibrium point. A standard procedure to describe the dynamics in these cases is to generate a so–called *incremental model*—whose equilibrium is at zero and with inputs and outputs the deviations with respect to their value at the equilibrium. A natural question that arises is whether a property of the original system is inherited by its incremental model. In this paper, we explore this question regarding passivity. More specifically, we provide a solution to the following problem.

(Passivity of Incremental Systems) Given a nonlinear system of the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{g}\mathbf{u} \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}), \end{aligned} \tag{1}$$

where $\mathbf{x}, \mathbf{u}, \mathbf{y}$ are functions of t, $\mathbf{x}(t) \in \mathbb{R}^n, \mathbf{u}(t), \mathbf{y}(t) \in \mathbb{R}^m$, with $m \leq n$, the functions $\mathbf{f}, \mathbf{h}$ are locally Lipschitz and the matrix $\mathbf{g} \in \mathbb{R}^{n\times m}$ is constant and has full rank. Define the incremental model

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{x}^\star) - \mathbf{f}(\mathbf{x}^\star) + \mathbf{g}\tilde{\mathbf{u}} \\ \tilde{\mathbf{y}} &= \mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{x}^\star), \end{aligned} \tag{2}$$

where $\tilde{(\cdot)} = (\cdot) - (\cdot)^\star$ are the incremental variables, $\mathbf{x}^\star \in \mathbb{R}^n$ is an equilibrium point, that is,

$$\mathbf{x}^\star \in \mathcal{E} := \{\bar{\mathbf{x}} \in \mathbb{R}^n \mid \mathbf{g}^\perp \mathbf{f}(\bar{\mathbf{x}}) = 0\}, \tag{3}$$

where $\mathbf{g}^\perp \in \mathbb{R}^{(n-m)\times n}$ is a full–rank left–annihilator of $\mathbf{g}$, i.e., $\mathbf{g}^\perp \mathbf{g} = 0$ and rank$\{\mathbf{g}^\perp\} = n - m$, and $\mathbf{u}^\star, \mathbf{y}^\star \in \mathbb{R}^m$ are the constant input and output vectors associated to the equilibrium state $\mathbf{x}^\star$, that is

$$\begin{aligned} \mathbf{u}^\star &= (\mathbf{g}^\top \mathbf{g})^{-1} \mathbf{g}^\top \mathbf{f}(\mathbf{x}^\star) \\ \mathbf{y}^\star &= \mathbf{h}(\mathbf{x}^\star). \end{aligned} \tag{4}$$

Assume (1) defines a passive mapping $\mathbf{u} \to \mathbf{y}$. Under which conditions the mapping $\tilde{\mathbf{u}} \to \tilde{\mathbf{y}}$, defined by (2), is also passive?

The *main contributions* of this paper are, first, the establishment of a condition on the vector field $\mathbf{f}(\mathbf{x})$ to ensure passivity of the mapping $\tilde{\mathbf{u}} \to \tilde{\mathbf{y}}$. Second, we prove that a large class of nonlinear RLC circuits—with convex electric and magnetic energy functions and passive resistors with monotonic characteristic functions—satisfy this condition, hence showing that these circuits can be globally stabilized with linear PI control.

II. SOME COMMENTS AND MOTIVATION

1. The question posed above can be recast without invoking incremental models, but using the more general concept of dissipative systems [11], as follows. Assume (1) is dissipative with supply rate $\mathbf{u}^\top \mathbf{y}$, (this is, of course, equivalent to passivity of the mapping $\mathbf{u} \to \mathbf{y}$). Under which conditions (1) is also dissipative with respect to the incremental supply rate $\tilde{\mathbf{u}}^\top \tilde{\mathbf{y}}$? In view of the ubiquity of incremental models in applications we have opted for the formulation of the problem given above.

2. Invoking Kalman–Yakubovich–Popov's Lemma [10] it is easy to establish that all passive linear time invariant (LTI) systems have passive incremental models. Indeed, if $H(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{P}\mathbf{x}$, with $\mathbf{P} \in \mathbb{R}^{n\times n}, \mathbf{P} = \mathbf{P}^\top > 0$, is a storage function for the original system, $H(\tilde{\mathbf{x}}) = \frac{1}{2}\tilde{\mathbf{x}}^\top \mathbf{P}\tilde{\mathbf{x}}$ is a storage function for the incremental model as well.

3. Passivity of incremental models has been explored in [6] for the case when (1) is a port controlled Hamiltonian system [10]. Actually, the storage function constructed here is the one used in [6]—but expressed in the original coordinates of the system, see Remark 1.

4. Motivations to establish passivity of incremental models are manifold. It has been used in [5] for tracking and disturbance rejection—via internal model principles—in passive systems. Another immediate application concerns

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energy–balancing stabilization. As defined in [9] a system is energy–balancing stabilizable if there exists a static state–feedback that assigns to the closed–loop system a storage function equal to the difference between the (open–loop) systems stored energy and the energy supplied by the controller, i.e., $\int \mathbf{u}^\top(s)\mathbf{y}(s)ds$. As indicated in [9], see also [8], energy–balancing stabilization is stymied by the presence of pervasive dissipation. The latter is defined as dissipation that makes the supplied power evaluated at the equilibrium non–zero, that is $(\mathbf{u}^\star)^\top \mathbf{y}^\star \neq 0$. It is clear that this obstacle is conspicuous by its absence in systems with passive incremental models. Results stemming from this observation will be reported elsewhere.

5. We have adopted in the paper the standard convention of defining passive systems in terms of the existence of a *non–negative* storage function.[1] As will become clear below, all our derivations remain valid if we relax the non–negativity assumption. These, obviously larger, class of systems are referred in [8] as energy–balancing and in [12], [4] as cyclo–passive—a name that is motivated by the fact that cyclo–passive systems cannot create energy *over closed paths* in the state–space, in contrast with passive system that cannot create energy *for all* trajectories.

III. PASSIVITY OF INCREMENTAL SYSTEMS

Proposition 1: Assume:

A.1 The system (1) defines a passive mapping $\mathbf{u} \rightarrow \mathbf{y}$ with a convex twice continuously differentiable storage function $H : \mathbb{R}^n \rightarrow \mathbb{R}_+$.

A.2 The *"incremental stability" condition*

$$[\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^\star)]^\top [\nabla H(\mathbf{x}) - \nabla H(\mathbf{x}^\star)] \leq 0 \qquad (5)$$

is satisfied.[2]

Then, the mapping $\tilde{\mathbf{u}} \rightarrow \tilde{\mathbf{y}}$, defined by (2) is also passive with convex storage function $H_0 : \mathbb{R}^n \rightarrow \mathbb{R}_+$,

$$H_0(\tilde{\mathbf{x}}) = H(\tilde{\mathbf{x}} + \mathbf{x}^\star) - H(\mathbf{x}^\star) - \tilde{\mathbf{x}}^\top \nabla H(\mathbf{x}^\star). \qquad (6)$$

Proof: First, we recall from Hill–Moylan's nonlinear version of Kalman–Yakubovich–Popov's Lemma [10] that passivity of (1) implies

$$\mathbf{h}(\mathbf{x}) = \mathbf{g}^\top \nabla H(\mathbf{x}). \qquad (7)$$

Let us now compute

$$\begin{aligned} \tilde{\mathbf{y}}^\top \tilde{\mathbf{u}} &= [\nabla H(\mathbf{x}) - \nabla H(\mathbf{x}^\star)]^\top \mathbf{g}\tilde{\mathbf{u}} \\ &= [\nabla H(\mathbf{x}) - \nabla H(\mathbf{x}^\star)]^\top [\dot{\tilde{\mathbf{x}}} - \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{x}^\star) + \mathbf{f}(\mathbf{x}^\star)] \\ &= \dot{H}_0 - [\nabla H(\mathbf{x}) - \nabla H(\mathbf{x}^\star)]^\top [\mathbf{f}(\tilde{\mathbf{x}} + \mathbf{x}^\star) - \mathbf{f}(\mathbf{x}^\star)] \\ &\geq \dot{H}_0, \end{aligned}$$

where we have used (7) in the first identity, (2) in the second, (6) in the third and (5) to obtain the inequality. Integrating the inequality above we get

$$\int_0^t \tilde{\mathbf{y}}^\top(s)\tilde{\mathbf{u}}(s)\ \mathrm{d}s \geq H_0(\tilde{\mathbf{x}}(t)) - H_0(\tilde{\mathbf{x}}(0)).$$

We will now prove that $H_0(\tilde{\mathbf{x}})$ is nonnegative. Using convexity of $H(\mathbf{x})$, we obtain

$$\frac{\partial^2 H_0}{\partial \tilde{\mathbf{x}}^2}(\tilde{\mathbf{x}}) = \frac{\partial^2 H}{\partial \mathbf{x}^2}(\mathbf{x}) \geq 0,$$

which shows the convexity of $H_0(\tilde{\mathbf{x}})$. Since $\nabla H_0(0) = 0$ and $H_0(\tilde{\mathbf{x}})$ is convex, the point 0 is a minimum point of $H_0(\tilde{\mathbf{x}})$. This implies also that $H_0(\tilde{\mathbf{x}}) \geq 0$. ■

Remark 1: The storage function $H_0(\tilde{\mathbf{x}})$ can be directly derived from [6]—where the case of port Hamiltonian systems is considered and the analysis is carried out in co–energy coordinates ($\nabla H(\mathbf{x})$ in our notation). Indeed, integrating equation (10) from that paper and expressing the function in the original (energy) coordinates, denoted $\mathbf{x}$ here and called $\mathbf{z}$ in [6], yields (6).

Remark 2: Passivity of (1) imposes, besides (7), the stability condition $\mathbf{f}^\top(\mathbf{x})\nabla H(\mathbf{x}) \leq 0$. This motivates the name "incremental stability" given to inequality (5).

Remark 3: In the LTI case with quadratic storage function (5) reduces to the stability condition $\tilde{\mathbf{x}}^\top \mathbf{P}\mathbf{A}\tilde{\mathbf{x}} \leq 0$, while the new storage function is given by $H_0(\tilde{\mathbf{x}}) = \frac{1}{2}\tilde{\mathbf{x}}^\top \mathbf{P}\tilde{\mathbf{x}}$. This appealing downward compatibility makes our result a natural extension, to the nonlinear case, of the well–known property of LTI systems.

Remark 4: If (1) is a port controlled Hamiltonian system we have

$$\mathbf{f}(\mathbf{x}) = [\mathbf{J}(\mathbf{x}) - \mathbf{R}(\mathbf{x})]\nabla H(\mathbf{x}),$$

where $\mathbf{J}(\mathbf{x}) = -\mathbf{J}^\top(\mathbf{x})$ is the interconnection matrix and $\mathbf{R}(\mathbf{x}) = \mathbf{R}^\top(\mathbf{x}) \geq 0$ captures the dissipation effects. The incremental stability condition (5) will be then satisfied if $\mathbf{J}$ and $\mathbf{R}$ are *constant* matrices. This corresponds to constant interconnections and linear damping—the former is often the case in physical systems, for instance for nonlinear mechanical systems or nonlinear LC circuits. In the next section we will prove that the incremental model of passive RLC circuits is passive also in the case when the resistors are nonlinear, but with monotonic characteristic function.

Remark 5: For port–controlled Hamiltonian systems with constant interconnection and damping matrices the storage function for the incremental model given in (6) results from a direct application of the Interconnection and Damping Assignment controller design methodology [9]. Indeed, in its simpler formulation, the objective of this controller is to shape the storage function of the system assigning to the closed–loop the dynamics $\dot{\mathbf{x}} = (\mathbf{J} - \mathbf{R})\nabla H_d(\mathbf{x}) + \mathbf{g}\mathbf{v}$, where $H_d(\mathbf{x})$ is the desired storage function and $\mathbf{v}$ is a free

[1] Actually, as one can always add a constant to the storage function, the question is whether it is bounded from below or not.

[2] All vectors defined in the paper are *column* vectors, even the gradient of a scalar function that we denote with the operator $\nabla_{\mathbf{x}} = \partial/\partial\mathbf{x}$. When clear from the context the sub-index will be omitted.

external signal. *Fixing* $\mathbf{v} = \tilde{\mathbf{u}}$ we see that the objective will be achieved with $H_d(\mathbf{x}) = H_0(\mathbf{x})$; and by definition of the equilibrium set (3), the matching equation

$$-\mathbf{g}\mathbf{u}^\star = (\mathbf{J} - \mathbf{R})\nabla H(\mathbf{x}^\star),$$

always has a solution.

It is easy to see that the new storage function H_0 as in (6) has a minimum at 0 and $H_0(0) = 0$. However, the point 0 may not be a unique minimum. The following lemma shows that a strongly convex H is sufficient to ensure that the new storage function H_0 has a unique minimum at 0 and it is proper. By properness of the function H_0 we mean that for any constant $c > 0$ the set $\{\mathbf{x} \in \mathbb{R}^n \mid H_0(\mathbf{x}) \le c\}$ is compact.

Lemma 1: Consider a strongly convex storage function $H \in \mathcal{C}^2$. Then the function $H_0 \in \mathcal{C}^2$ as in (6) is proper and has a unique minimum at 0.

Proof: It can be readily checked that H_0, being globally strictly convex, has a unique global minimum at the origin.

In order to prove properness of H_0, we demonstrate that for every $c \ge 0$ the set $M_c = \{\tilde{\mathbf{x}} \in \mathbb{R}^n \mid H_0(\tilde{\mathbf{x}}) \le c\}$ is compact. Strong convexity of H_0 implies that the sets M_c are bounded [1, p. 460] and continuity implies that they are closed [2, p.17]. Since the sets M_c have a finite dimension, it follows that they are compact. ∎

IV. PI STABILIZATION OF NONLINEAR RLC CIRCUITS

In this section, we prove that a large class of nonlinear RLC circuits—with convex electric and magnetic energy functions and passive resistors with monotonic characteristic functions—satisfy the condition of Proposition 1, hence showing that these circuits can be globally stabilized with linear PI control.

We consider RLC circuits consisting of interconnections of (possibly nonlinear) lumped dynamic (n_L inductors, n_C capacitors) and static (n_R resistors, n_{v_S} voltage sources and n_{i_S} current sources) elements. Capacitors and inductors are defined by the physical laws and constitutive relations [3]:

$$\mathbf{i}_C = \dot{\mathbf{q}}_C, \quad \mathbf{v}_C = \nabla H_C(\mathbf{q}_C), \tag{8}$$

$$\mathbf{v}_L = \dot{\phi}_L, \quad \mathbf{i}_L = \nabla H_L(\phi_L), \tag{9}$$

respectively, where $\mathbf{i}_C(t)$, $\mathbf{v}_C(t)$, $\mathbf{q}_C(t) \in \mathbb{R}^{n_C}$ are the capacitors currents, voltages and charges, and $\mathbf{i}_L(t)$, $\mathbf{v}_L(t)$, $\phi_L(t) \in \mathbb{R}^{n_L}$ are the inductors current, voltage and flux–linkages, $H_L : \mathbb{R}^{n_L} \to \mathbb{R}$ is the magnetic energy stored in the inductors and $H_C : \mathbb{R}^{n_C} \to \mathbb{R}$ is the electric energy stored in the capacitors. We also define the total energy as

$$H(\phi_L, \mathbf{q}_C) = H_L(\phi_L) + H_C(\mathbf{q}_C).$$

To avoid cluttering (even more) the notation, and without loss of generality, we will consider that all current (resp. voltage) controlled resistors are in series with inductors (resp. in parallel with capacitors). In this way, we can write $\mathbf{v}_{R_{L_i}} = \hat{\mathbf{v}}_{R_{L_i}}(\mathbf{i}_{L_i})$ for the current controlled resistors and

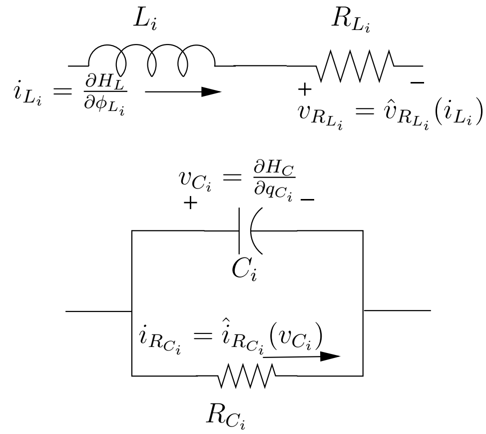

Fig. 1. Current controlled resistors in series with inductors and voltage controlled resistors in parallel with capacitors

$\mathbf{i}_{R_{C_i}} = \hat{\mathbf{i}}_{R_{C_i}}(\mathbf{v}_{C_i})$ for the voltage controlled resistors, where $\hat{\mathbf{v}}_{R_{L_i}}, \hat{\mathbf{i}}_{R_{C_i}} : \mathbb{R} \to \mathbb{R}$ are their characteristic curves. See Fig. 1

The dynamics of the circuit can be written as a slight extension—to the case of nonlinear resistors—of the port–controlled Hamiltonian model of LC circuits described in [7][3]

$$\begin{bmatrix} \dot{\phi}_L \\ \dot{\mathbf{q}}_C \end{bmatrix} = \mathbf{J}\,\nabla H - \begin{bmatrix} \hat{\mathbf{v}}_{R_L}(\nabla H_L(\phi_L)) \\ \hat{\mathbf{i}}_{R_C}(\nabla H_C(\mathbf{q}_C)) \end{bmatrix} + \mathbf{g}\,\mathbf{u} \tag{10}$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & -\mathbf{\Gamma} \\ \mathbf{\Gamma}^\top & 0 \end{bmatrix},\ \mathbf{g} = \begin{bmatrix} -\mathbf{B}_{v_S} & 0 \\ 0 & \mathbf{B}_{i_S} \end{bmatrix},\ \mathbf{u} = \begin{bmatrix} \mathbf{v}_{v_S} \\ \mathbf{i}_{i_S} \end{bmatrix}.$$

$\mathbf{v}_{v_S}(t) \in \mathbb{R}^{n_{v_S}}$ are the voltage sources (in series with inductors), $\mathbf{i}_{i_S}(t) \in \mathbb{R}^{n_{i_S}}$ the current sources (in parallel with capacitors), $\mathbf{B}_{v_S} \in \mathbb{R}^{n_L \times n_{v_S}}$, $\mathbf{B}_{i_S} \in \mathbb{R}^{n_C \times n_{i_S}}$ are input (full rank) matrices with $n_{v_S} \le n_L$, $n_{i_S} \le n_C$ and $\mathbf{\Gamma} \in \mathbb{R}^{n_L \times n_C}$, is a constant matrix determined by the circuit topology.

The port variables are completed defining the currents and voltages of the sources, which are given by

$$\mathbf{y} = \mathbf{g}^\top \nabla H = \begin{bmatrix} -\mathbf{B}_{v_S}^\top \nabla H_L(\phi_L) \\ \mathbf{B}_{i_S}^\top \nabla H_C(\mathbf{q}_C) \end{bmatrix} \tag{11}$$

Proposition 2: Consider the dynamics of the nonlinear RLC circuit (10), (11). Let $(\phi_{L\star}, \mathbf{q}_{C\star})$ be an equilibrium point with the corresponding constant input $\mathbf{u}^\star$ and output $\mathbf{y}^\star$. Assume

B.1 Inductors and capacitors are passive and their energy functions are twice continuously differentiable and strongly convex.

B.2 The resistors are passive and their characteristic functions are monotone non–decreasing.

Then, the circuit in closed–loop with the PI controller

$$\begin{aligned} \dot{\xi} &= -\tilde{\mathbf{y}} \\ \mathbf{u} &= K_I \xi - K_P \tilde{\mathbf{y}} \end{aligned} \tag{12}$$

[3]Notice that if the resistors are linear, equation (10) takes the more familiar form $\dot{\mathbf{x}} = [\mathbf{J} - \mathbf{R}]\nabla H + \mathbf{g}\mathbf{u}$ [10].

where $K_I = K_I^\top > 0,\ K_P = K_P^\top > 0$, ensures all state trajectories $(\phi_L(t), \mathbf{q}_C(t), \xi(t))$ are bounded and

$$\lim_{t\to\infty} \|\tilde{\mathbf{y}}(t)\| = 0.$$

If, in addition, the closed loop system (10), (11), (12) satisfies the detectability assumption

B.3

$$\tilde{\mathbf{y}}(t) \equiv 0, \tilde{\mathbf{u}}(t) \equiv 0 \ \Rightarrow \lim_{t\to\infty} \left\| \begin{bmatrix} \tilde{\phi}_L(t) \\ \tilde{\mathbf{q}}_C(t) \\ \tilde{\xi}(t) \end{bmatrix} \right\| = 0,$$

where $\xi^\star = K_I^{-1}\mathbf{u}^\star$.

Then,

$$\lim_{t\to\infty} \left\| \begin{bmatrix} \tilde{\phi}_L(t) \\ \tilde{\mathbf{q}}_C(t) \\ \tilde{\xi}(t) \end{bmatrix} \right\| = 0.$$

Proof: First, invoking Proposition 1, we will prove that the incremental model of the circuit defines a passive system $\tilde{\mathbf{u}} \to \tilde{\mathbf{y}}$ with a *proper positive definite* storage function. Since the PI is a passive system, the proof will be then completed with standard passivity–based control arguments.

As is well–known RLC circuits with passive elements are passive [3] with storage function their total energy. Indeed, computing

$$\begin{array}{rcl} \dot{H}(\phi_L, \mathbf{q}_C) & = & -\mathbf{i}_L^\top \hat{\mathbf{v}}_{R_L}(\mathbf{i}_L) - \mathbf{v}_C^\top \hat{\mathbf{i}}_{R_C}(\mathbf{v}_C) + \mathbf{y}^\top \mathbf{u} \\ & \leq & \mathbf{y}^\top \mathbf{u} \end{array}$$

where we have used (8), (9) and (11) to get the identity and passivity of the resistors of Assumption B.2 to obtain the inequality.[4] Non–negativity of $H(\phi_L, \mathbf{q}_C)$ follows from passivity of inductors and capacitors of Assumption B.1.

To prove passivity of the incremental model of (10), (11) we need to verify the "incremental" stability condition (5) which after some calculations becomes

$$\begin{bmatrix} -\hat{\mathbf{v}}_{R_L}(\nabla H_L(\phi_L)) + \hat{\mathbf{v}}_{R_L}(\nabla H_L(\phi_L^\star)) \\ -\hat{\mathbf{i}}_{R_C}(\nabla H_C(\mathbf{q}_C)) + \hat{\mathbf{i}}_{R_C}(\nabla H_C(\mathbf{q}_C^\star)) \end{bmatrix}^\top \times$$

$$\begin{bmatrix} \nabla H_L(\phi_L) - \nabla H_L(\phi_L^\star) \\ \nabla H_C(\mathbf{q}_C) - \nabla H_L(\mathbf{q}_C^\star) \end{bmatrix} =$$

$$\begin{array}{rl} = & -(\hat{\mathbf{v}}_{R_L}(\mathbf{i}_L) - \hat{\mathbf{v}}_{R_L}(\mathbf{i}_L^\star))^\top (\mathbf{i}_L - \mathbf{i}_L^\star) \\ & -(\hat{\mathbf{i}}_{R_C}(\mathbf{v}_C) - \hat{\mathbf{i}}_{R_C}(\mathbf{v}_C^\star))^\top (\mathbf{v}_C - \mathbf{v}_C^\star) \\ \leq & 0, \end{array}$$

where we have used equations (8) and (9) for the first identity and the monotonic resistors characteristic condition of Assumption B.2 for the inequality.

The storage function for the incremental model is computed from (6) as

$$\begin{array}{rcl} H_0(\tilde{\phi}_L, \tilde{\mathbf{q}}_C) & = & H_L(\tilde{\phi}_L + \phi_L^\star) + H_C(\tilde{\mathbf{q}}_C + \mathbf{q}_C^\star) - H_L(\phi_L^\star) \\ & & -H_C(\mathbf{q}_C^\star) - \tilde{\phi}_L^\top \nabla H_L(\phi_L^\star) - \tilde{\mathbf{q}}_C^\top \nabla H_C(\mathbf{q}_C^\star) \end{array} \quad (13)$$

[4] We recall that resistors are passive if and only if their characteristic function lives in the first–third quadrant [3].

which, under Assumption B.1, is strongly convex and has a global minimum at the origin.

According to Lemma 1, the function H_0 is proper and has a unique global minimum at 0.

To complete the proof of the proposition we note that the incremental model of the closed–loop system takes the form

$$\dot{z} = F(z), \quad \tilde{y} = H(z),$$

where $z = \mathrm{col}(\tilde{\phi}_L, \tilde{\mathbf{q}}_C, \tilde{\xi})$ and $F(z), H(z)$ are continuous. We, thus, consider the (positive definite and proper) Lyapunov function candidate

$$H_{cl}(z) = H_0(\tilde{\phi}_L, \tilde{\mathbf{q}}_C) + \frac{1}{2}\tilde{\xi}^\top K_I \tilde{\xi}.$$

Computing the derivative

$$\begin{array}{rcl} \dot{H}_{cl} & \leq & \tilde{\mathbf{y}}^\top \tilde{\mathbf{u}} - \xi^\top K_I \tilde{\mathbf{y}} \\ & = & -\tilde{\mathbf{y}}^\top K_P \tilde{\mathbf{y}}. \end{array} \quad (14)$$

It follows from (14) that the state $z(t)$ is bounded and $\tilde{\mathbf{y}}(t)$ is square integrable. From continuity of $F(z)$, this also implies that $\dot{z}(t)$ is bounded, hence $z(t)$ is uniformly continuous. From continuity of $H(z)$ we also have that $\tilde{\mathbf{y}}(t)$ is uniformly continuous, and we conclude $\lim_{t\to\infty} \|\tilde{\mathbf{y}}(t)\| = 0$.

Convergence of the incremental state to zero follows using LaSalle's invariance principle and invoking the zero–state detectability of Assumption B.3 (see, for example, [10]). ∎

V. CONCLUSIONS AND FUTURE RESEARCH

We have considered general affine passive systems with constant input matrix. We defined an "incremental stability" condition on the vector field $\mathbf{f}(\mathbf{x})$ that ensures passivity of the incremental model. Then, we showed that a large class of nonlinear passive RLC circuits—with convex and proper electric and magnetic energy functions and monotonic resistor characteristics—satisfy this condition. Hence, these circuits can be globally stabilized with linear PI control.

Current research is under way along two directions. First, to employ these results for energy–balancing stabilization of physical systems. Second, to derive conditions for passivity of more general error models, for instance, those that appear when tracking feasible trajectories.

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