Robust Finite-time Stabilisation of an Arbitrary-Order Nonholonomic System in Chained Form ⋆

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Abstract

We present a robust finite-time stabilising algorithm for an arbitrary-order nonholonomic system in chained form. The state feedback is locally bounded, homogeneous and discontinuous. Its design follows from a generalised circle criterion based on the theory of homogeneous Lyapunov functions and a two-stage homogeneity-preserving strategy proposed by Nakamura.

Key words: Absolute stability; Homogeneous systems; Nonholonomic systems; Chained form; Finite-time stabilisation.

1 Introduction

The problems of stabilisation and control of nonholonomic systems have attracted research attention for several decades. Typical applications include multi-fingered hands and mobile and space robots [1]. The main theoretical challenge is that, even though such systems are controllable, they fail Brockett's test [2] and hence are not stabilisable by continuous static feedback laws.

There exists an extensive literature on control problems for nonholonomic systems (see [3] for a survey), although the great majority focuses on unperturbed nominal models. The absence of continuous time-invariant state-feedback asymptotic stabilisers has oriented research towards two main stabilisation techniques: by continuous but time-varying laws, and by time-invariant but discontinuous laws. Regarding time-varying feedback laws, the following dichotomy cannot be avoided [4]:

- Differentiable or at least Lipschitz-continuous asymptotic stabilisers can be achieved. These yield good robustness properties and low noise sensitivity. However, they have slower-than-exponential convergence to the origin (see, for instance, [5–9]).
- Uniform exponential stability can be obtained with Hölder continuous feedback laws, but such controllers have shown to lack robustness and exhibit high sensitivity to noise [7] [1].

The aforementioned undesirable features of time-varying feedback laws motivate the research on discontinuous time-invariant stabilisers. However, other problems can arise due to the presence of discontinuities. For instance, some control laws are locally unbounded [10]. That is, the controller takes arbitrarily large values for states sufficiently close to a switching surface. A controller with this feature comes with obvious implementation issues. A discontinuous locally bounded stabiliser can be found in [11]. The authors propose a state feedback law that globally exponentially stabilises a class of nonholonomic systems. However, it is not specified in which sense the solutions are considered (Carathéodory, Filippov or some other), which fails to provide an accurate description of some of the phenomena arising from the discontinuous nature of the vector field (for example, when trajectories approach the surface of discontinuity from both sides).

Many mechanical systems with nonholonomic constraints can be written in the well-known chained or power forms [1,12–16]. When written in these forms, the Lie algebras generated by the system vector fields

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[1] In the context given in [7], robustness refers to "the property of preserving the closed-loop stability of the desired equilibrium against small structured modelling errors, control delays, fluctuations of the sampling period, etc."

are nilpotent, the controllability rank condition is easily verified, and the failure of Brockett's test is readily exposed. In this paper, we also consider the problem of stabilising n-dimensional systems in chained form, but we take uncertainties into account.

In the sliding-mode control literature, one finds controllers achieving finite-time practical stabilisation [17–19] with additional robustness properties. The integral sliding mode in [20] uses a dynamic extension to find a discontinuous feedback law that stabilises the so-called extended Heisenberg system. The system can be transformed into the chained form with additional integrators. Finally, we mention that the robust stabilisation problem for chained systems can be approached from an adaptive-control perspective [21,22], but the approach only works for structured uncertainty.

We present a static state-feedback controller that almost globally asymptotically stabilises the origin of an nth-order system in chained form. Distinctive features of the controller are: (i) robustness in the form of absolute stability with respect to a homogeneous sector, (ii) finite-time convergence for almost every initial condition, and (iii) local boundedness. We elaborate on the two-stage homogeneity-preserving controller presented in [23]. By imposing a negative degree, finite-time stability is attained. Likewise, homogeneity lends itself to the application of the generalised circle criterion [24] in order to achieve absolute stability. An explicit homogeneous control Lyapunov function is instrumental in the design of the controller.

This article is organised as follows: In Section 2 we introduce some background concepts about finite-time stability, homogeneity, and the homogeneous generalisation of the circle criterion. Additionally, the objectives of this work will be detailed in the problem statement. Section 3 presents the proposed stabilisation law and a proof of its robustness. Simulations support the analysis. Finally, the conclusions are given in Section 4.

2 Preliminaries and Problem Statement

Notation

We denote the set of nonnegative real numbers by $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$. For a scalar s, $|s|$ represents its absolute value, and the notation $\lceil s \rfloor^p$ stands for $|s|^p \operatorname{sign}(s)$. For a vector v, $\|v\|$ denotes its Euclidean norm and v_i its ith coordinate. For a set S, its interior and boundary are designated by $\mathring{S}$ and ∂S, respectively, and the set difference of S and another set N is written as $S \setminus N$. A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\sigma(0) = 0$ and the function is strictly increasing. Given a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, we denote the partial derivative of V with respect to x_1 by $\nabla_{x_1} V$ and the row vector of n partial derivatives by ∇V.

2.1 Finite-Time Stability

Consider the system

$$\dot{x} = f(t, x) , \quad t \in \mathbb{R}_+ , \quad x(0) = x_0 , \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector. The nonlinear vector field $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be locally bounded uniformly in time. When f is locally measurable but discontinuous with respect to x, the solutions are meant in the sense of Filippov [25]. That is, $x(t, x_0)$ is a solution if it is absolutely continuous and if it satisfies the differential inclusion

$$\dot{x} \in K[f](t, x) = \operatorname{co} \bigcap_{\varepsilon > 0} \bigcap_{\mu(N) = 0} f(t, B(x, \varepsilon) \setminus N) , \tag{2}$$

where $\operatorname{co}(M)$ stands for the convex closure of the set M, $B(x, \varepsilon)$ represents the ball centred at x with radius ε, and the equality $\mu(N) = 0$ indicates that the set $N \subset \mathbb{R}^n$ has zero Lebesgue measure.

Definition 1 ([26]) *Let the origin be an equilibrium of* (1)*, i.e., let* $0 \in K[f](t, 0)$*. We say that it is* globally finite-time stable *if it satisfies the following properties:*

i) Lyapunov stability: *there exists a function* $\alpha \in \mathcal{K}$ *such that* $\|x(t, x_0)\| \leq \alpha(\|x_0\|)$.
ii) Finite-time attractivity: *there exists a locally bounded function* $T : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_+$ *such that* $x(t, x_0) = 0$ *for all* $t \geq T(x_0)$*. Such a function* T *is called a* settling-time function.

2.2 Homogeneity

Given $\varepsilon > 0$ and a vector of weights $r = (r_1 \; r_2 \; \cdots \; r_n)^\top$ with $0 < r_j < \infty$, $j = 1, 2, \ldots, n$, we define the *dilation* operator $\delta_\varepsilon^r : \mathbb{R}^n \to \mathbb{R}^n$ as $\delta_\varepsilon^r(x) := (\varepsilon^{r_1} x_1 \; \cdots \; \varepsilon^{r_n} x_n)^\top$.

A function $V : \mathbb{R}^n \to \mathbb{R}$ (respectively, a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$) is called *r-homogeneous* of degree $m \in \mathbb{R}$ (resp., $\tau \in \mathbb{R}$) if, for all $x \in \mathbb{R}^n$ and all $\varepsilon > 0$, the following identity holds: $V(\delta_\varepsilon^r(x)) = \varepsilon^m V(x)$ (resp., $f(\delta_\varepsilon^r(x)) = \varepsilon^\tau \delta_\varepsilon^r(f(x))$).

These definitions correspond to the concept of *weighted homogeneity*, as introduced in [27–29]. When clear from context, we will not explicitly mention r. Given $p \geq 1$, the homogeneous p-norm is defined as $\|x\|_{r,p}^p := \sum_{j=1}^n |x_j|^{\frac{p}{r_j}}$. We also define the r-homogeneous sphere

$$\mathbb{S}_{r,p} = \{x \in \mathbb{R}^n \mid \|x\|_{r,p} = 1\} .$$

If V is continuous and homogeneous of degree m, $\deg V = m$, it easy to show the existence of constants $C_1, C_2 > 0$ such that

$$C_1 \|x\|_{r,p}^m \leq V(x) \leq C_2 \|x\|_{r,p}^m . \tag{3}$$

Homogeneous Lyapunov functions are a useful tool for the stability analysis of homogeneous systems, as suggested by the following converse theorem.

Theorem 2 ([28]) *Consider a continuous vector field $f : \mathbb{R}^n \to \mathbb{R}^n$. Let f be r-homogeneous of degree τ, and let the origin of $\dot{x} = f(x)$ be locally asymptotically stable. Then, the origin is globally asymptotically stable. Moreover, for any m such that*

$$m > \max_{1 \le j \le n} r_j \ , \tag{4}$$

there exists a continuously differentiable Lyapunov function V which is r-homogeneous of degree m.

If, furthermore, $\tau < 0$, the system $\dot{x} = f(x)$ is globally finite-time stable, and the settling time is continuous at zero and locally bounded.

The existence of an r-homogeneous continuously-differentiable Lyapunov function V can be exploited to establish the robustness of an r-homogeneous vector field.

Corollary 3 *Consider a continuous, asymptotically stable, and r-homogeneous vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, and let $\Delta f : \mathbb{R}^n \to \mathbb{R}^n$ be another continuous vector field. Choose any $p \in \mathbb{N}$ and assume that, for $i = 1, \dots, n$, either:*

- *i)* $\Delta f_i(\delta_\varepsilon^r(x)) = o(\varepsilon^{\tau + r_i})$ *as* $\varepsilon \to 0$,
- *ii)* $\Delta f_i(\delta_\varepsilon^r(x)) = O(\varepsilon^{\tau + r_i})$ *as* $\varepsilon \to 0$,
- *iii)* $\Delta f_i(\delta_\varepsilon^r(x)) + c_i = O(\varepsilon^{\tau + r_i})$ *as* $\varepsilon \to 0$, $c_i \in \mathbb{R}$,

uniformly with respect to x on the sphere $\mathbb{S}_{r,p}$. (Here, 'o' and 'O' refer to Landau's little-o and big-o notations.) Then, respectively,

- *i) The origin of $f + \Delta f$ is still asymptotically stable,*
- *ii) The origin of $f + \Delta f \cdot q$ is still asymptotically stable for $q \in \mathbb{R}$ with $|q|$ small enough,*
- *iii) The trajectories of $f + \Delta f \cdot q$ are ultimately bounded for $q \in \mathbb{R}$ with $|q|$ small enough.*

Item i) has been proved in [28]. Items ii) and iii) are straightforward extensions that result from the following crucial property: If V is of degree m, then $\partial V / \partial x_i$ is r-homogeneous of degree $m - r_i$.

We will devote the rest of the paper to a stronger notion of stability: absolute stability. Absolute stability is stronger than the items of Corollary 3 in that the global character of asymptotical stability is preserved.

2.3 Generalised Circle Criterion

The problem of *absolute stability* consists in finding conditions for which the origin of a closed-loop system is asymptotically stable, regardless of any output feedback belonging to a specific class. In particular, the problem of Lur'e refers to the case in which the system is linear, and the output feedback is memoryless but possibly time-varying. The circle criterion provides a solution to this particular case and is a typical example of the use of linear tools in the analysis of nonlinear systems.

In [24], a generalised circle criterion is introduced for Lur'e-like systems in which the system in the forward path is homogeneous. Recently developed tools for homogeneous systems allow us to achieve more ambitious objectives, such as finite-time stabilisation, and to address more general classes of systems.

Definition 4 (Homogeneous sector) *Given $k_1 \in \mathbb{R}$ and a non-zero r-homogeneous function $\phi : \mathbb{R}^n \to \mathbb{R}$, we say that the function $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ belongs to the r-homogeneous sector $[k_1, \infty)_\phi$ if it satisfies*

$$\phi(x)\left(\psi(t, x) - k_1 \phi(x)\right) \ge 0 \tag{5}$$

for all $t \in \mathbb{R}_+$ and all $x \in \mathbb{R}^n$.

For concreteness, we focus on unbounded homogeneous sectors, but it is straightforward to adapt the results of this paper to homogeneous sectors of the form $[k_1, k_2]$ with $k_1 < k_2 < +\infty$.

Definition 5 (Homogeneous absolute stability) *Consider a pair of continuous r-homogeneous vector fields $f, g : \mathbb{R}^n \to \mathbb{R}^n$, and an r-homogeneous function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\deg f = \deg g + \deg \phi$. The non-homogeneous and time-varying closed-loop system*

$$\dot{x} = f(x) - g(x)\psi(t, x) \tag{6}$$

is said to be absolutely stable with respect to $[k_1, \infty)_\phi$ *if the origin is a globally asymptotically stable equilibrium for any function ψ that belongs to the homogeneous sector $[k_1, \infty)_\phi$.*

Proposition 6 ([24]) *The closed-loop system* (6) *is absolutely stable if there exist a continuously differentiable r-homogeneous positive-definite function V of degree m, with m as in* (4)*, a positive-semidefinite function M and a positive-definite function L, both r-homogeneous, such that*

$$\nabla V(x)\left(f(x) - k_1 g(x)\phi(x)\right) = -L(x) \tag{7a}$$

$$\nabla V(x) g(x) = M(x)\phi(x) \ . \tag{7b}$$

Indeed, it is easy to verify that V is a Lyapunov function satisfying $\dot{V}(x) \le -L(x)$ for any ψ in the sector.

Conditions (7) with $M(x) = 1$ are a particular case of the well-known Hill-Moylan conditions for the passivity of nonlinear systems. The case $M(x) \neq 1$ releases V from requiring a fixed degree of homogeneity. This freedom will be exploited in the sequel, but may also prove useful in the context of other problems for which Lyapunov functions have to be designed.

2.4 Problem Statement

By applying coordinate and input transformations, nonholonomic systems can be written in canonical forms such as the so-called chained form [30]. For $n \geq 3$, a perturbed version of this form admits the representation

$$\begin{aligned} \dot{x}_1 &= u_1 + d_1(t,x) && \text{(8a)} \\ \dot{x}_j &= u_1 x_{j+1}\ , \quad j = 2,\dots,n-1 && \text{(8b)} \\ \dot{x}_n &= u_2 + d_2(t,x)\ . && \text{(8c)} \end{aligned}$$

Here, $x \in \mathbb{R}^n$ represents the state, $u_1, u_2 \in \mathbb{R}$ stand for the controls and $d_1, d_2 \in \mathbb{R}$ for the uncertainties or perturbations.

Assumption 7 *There exists a real constant $\bar{d}_1$ such that $|d_1(t,x)| \leq \bar{d}_1$.*

The features of d_2 will be discussed later in the article (see Assumption 14, below).

Recall that, even in the nominal case, i.e., when $d_1(t,x) = d_2(t,x) \equiv 0$, system (8) does not admit an asymptotically stabilising continuous state feedback, as the Brockett test shows. Moreover, because the system has nonholonomic constraints, the effect of static discontinuous feedback laws is also limited: if the solutions are intended in the sense of Filippov, then no locally bounded feedback can locally asymptotically stabilise the origin [31]. For further comments and examples on this result, we refer the reader to [32,28].

Recall that the basin of attraction of $x^\star \in \mathbb{R}^n$ is the set of all points x_0 such that the solution $x(t,x_0)$ is defined for all $t \geq 0$ and such that $\lim_{t\to\infty} x(t,x_0) = x^\star$.

Definition 8 ([33]) *The equilibrium $x = 0$ of a system $\dot{x} = f(t,x)$ is* almost globally asymptotically stable *if it is Lyapunov stable, and if the basin of attraction of the origin is a dense subset of the state space.*

Note that having this property is compatible with the results of [31], inasmuch as the basin of attraction is not necessarily a neighbourhood of the origin, so asymptotic stability is not claimed. In addition to this property, we will show that solutions starting in the basin of attraction converge to the origin in finite time, so that we actually provide almost global finite-time stability.

3 Robust Stabiliser

Following the main ideas in [23,34], a two-stage stabiliser is proposed for (8). First, the states x_j, $j = 2, \dots n$, are driven to zero in finite time while x_1 is driven away from zero. Next, x_1 is taken to zero, also in finite-time. The capability of robustly accomplishing these objectives relies on the homogeneous generalisation of the circle criterion presented above which, in turn, rests upon the choice of a homogeneous Lyapunov function. The resulting controls u_1, u_2 are easily-implementable locally bounded state-feedback laws.

3.1 Absolute Finite-Time Stability of a Chain of Integrators

Consider first a perturbed chain of $n-1$ integrators

$$\begin{aligned} \dot{\xi}_j &= \xi_{j+1}, \quad j = 1,\dots,n-2 \\ \dot{\xi}_{n-1} &= \upsilon + \delta(t,\xi)\ . \end{aligned} \tag{9}$$

We will construct a control law υ that stabilises the origin in finite time, regardless of any perturbation δ satisfying a particular homogeneous sector condition. In order to do so, we will apply the homogeneous circle criterion using the control Lyapunov function (CLF) given in [35] — Which in turn follows from the procedures presented in [36] and later in [34].

Proposition 9 ([35]) *Consider system (9) with $\delta(t,\xi) \equiv 0$. Choose any $0 > \tau > -1$, set the weights as $r_j = 1 - (n-1-j)\tau$, $j = 1,\dots,n-1$, and let $\{\beta_j\}$ be a non-decreasing sequence of $n-1$ positive real numbers such that*

$$r_1 \leq \beta_1 \leq \cdots \leq \beta_{n-2} \leq \beta_{n-1}\ . \tag{10}$$

Select a constant m such that $m \geq \max_{2\leq j\leq n-1} \beta_{j-1} + r_j$. Define the functions V_j recursively by

$$\begin{aligned} V_j(\xi) &= \gamma_{j-1} V_{j-1}(\xi) + W_j(\xi)\ , \quad \gamma_{j-1} > 0 \\ W_j(\xi) &= \frac{r_j}{m}|\xi_j|^{\frac{m}{r_j}} - \lceil w_{j-1}(\xi) \rfloor^{\frac{m-r_j}{r_j}} \xi_j \\ &\quad + \left(1 - \frac{r_j}{m}\right) |w_{j-1}(\xi)|^{\frac{m}{r_j}} \\ w_j(\xi) &= -\kappa_j \lceil \sigma_j(\xi) \rfloor^{\frac{r_{j+1}}{\beta_j}}\ , \quad \kappa_j > 0 \\ \sigma_j(\xi) &= \lceil \xi_j \rfloor^{\frac{\beta_j}{r_j}} - \lceil w_{j-1}(\xi) \rfloor^{\frac{\beta_j}{r_j}} \end{aligned} \tag{11}$$

and by the initial functions $V_1(\xi_1) = \frac{r_1}{m}|\xi_1|^{\frac{m}{r_1}}$, $\sigma_1(\xi) = \lceil \xi_1 \rfloor^{\frac{\beta_1}{r_1}}$, and $w_1(\xi) = -\kappa_1 \lceil \xi_1 \rfloor^{\frac{r_2}{r_1}}$ with $\kappa_1 > 0$.

Then, for sufficiently large values of $\kappa_j > 0$, $V = V_{n-1}$ is a CLF. More precisely, the continuous controller

$$\upsilon(\xi) = -\kappa \lceil \sigma_{n-1}(\xi) \rfloor^{\frac{1+\tau}{\beta_{n-1}}} \tag{12}$$

with $\kappa = \kappa_{n-1}$ implies that $\dot{V}(\xi) = -Z(\xi)$ with Z a positive-definite homogeneous function of degree $m + \tau$.

Note that σ_j, w_j and V_j are homogeneous of degrees β_j, r_{j+1} and m, respectively. Proposition 9 proves the stability of the closed-loop nominal system, and hence the existence of a class $\mathcal{K}$ function α_ξ such that

$$\|\xi(t, \xi_0)\| \leq \alpha_\xi(\|\xi_0\|) \quad t \geq 0 \ . \tag{13}$$

It also follows that

$$\dot{V}(\xi) \leq -\lambda V^{\frac{m+\tau}{m}}(\xi) \tag{14}$$

with $\lambda = \min_{\|x\|_{r,1}=1} Z(\xi)/V^{\frac{m+\tau}{m}}(\xi)$. Hence, the nominal chain of integrators is finite-time stable with the convergence time T^* readily upper-bounded by

$$T^*(\xi) \leq T(\xi) = -\frac{m}{\lambda \cdot \tau} V^{-\frac{\tau}{m}}(\xi) \ . \tag{15}$$

Analytically finding the set of gains κ_j in Proposition 9 is difficult, but a numerical approach can be found in [35]. Also, note that λ is proportional to κ_{n-1}.

Example 10 *Consider* (9) *with $n = 3$ and $\delta(t, \xi) \equiv 0$. For $\tau = -1/2$ we have $r_1 = 3/2$ and $r_2 = 1$. A simple choice satisfying* (10) *is $\beta_1 = \beta_2 = r_1$, and a simple choice for* (11) *is $\gamma_1 = 1$. The smallest m is then $5/2$. We have $V_1(\xi_1) = \frac{3}{5}|\xi_1|^{5/3}$. The virtual control $w_1(\xi_1) = -\kappa_1 \lceil \xi_1 \rfloor^{2/3}$ with gain $\kappa_1 = 1$ ensures that V_1 is a strict Lyapunov function for the subsystem $\dot{\xi}_1 = w_1(\xi_1)$. This leads to the function $V_2(\xi) = \frac{6}{5}|\xi_1|^{\frac{5}{3}} + \xi_1\xi_2 + \frac{2}{5}|\xi_2|^{\frac{5}{2}}$, which is positive definite on ξ_1, ξ_2. It also leads to the control $\upsilon(\xi) = -\kappa_2 \lceil \sigma_2(\xi) \rfloor^{\frac{1}{3}}$ with $\sigma_2(\xi) = \xi_1 + \lceil \xi_2 \rfloor^{\frac{3}{2}}$. The gain $\kappa_2 = 2$ ensures that $V = V_2$ is a strict Lyapunov function for* (9)*. That is, $\dot{V}_2(\xi) = -Z(x)$ with $Z(\xi) = \xi_2^2 - 2\xi_2(\xi_2 + \lceil \xi_1 \rfloor^{\frac{2}{3}}) + 2|\xi_1 + \lceil \xi_2 \rfloor^{\frac{3}{2}}|^{\frac{4}{3}}$ positive definite. Numerically solving for λ gives $\lambda = 0.94$.*

Proposition 11 *The chain of integrators* (9) *in closed-loop with* (12)*, but with $\kappa \geq \kappa_{n-1}$, is absolutely stable with respect to the homogeneous sector $[\kappa_{n-1} - \kappa, \infty)_{-\phi}$ with*

$$\phi(\xi) = \lceil \sigma_{n-1}(\xi) \rfloor^{\frac{1+\tau}{\beta_{n-1}}} \ . \tag{16}$$

PROOF. Set $\psi(t, \xi) = -\upsilon(\xi) - \delta(t, \xi)$ and $k_1 = \kappa_{n-1}$. System (9) takes the form (6) with $f(\xi) = (\xi_2 \ \xi_3 \ \cdots \ \xi_{n-2} \ 0)^\top$, $g(\xi) = (0 \ 0 \ \cdots \ 0 \ 1)^\top$ and the substitution $n \mapsto n - 1$.

Because the nominal system is stable with strict Lyapunov function V, condition (7a) holds for some positive definite r-homogeneous function L. Now, condition (7b) takes the form $\nabla_{\xi_{n-1}} W_{n-1}(\xi) = M(\xi)\phi(\xi)$. The left-hand side of this equality is

$$\nabla_{\xi_{n-1}} W_{n-1}(\xi) = \lceil \xi_{n-1} \rfloor^{\frac{m - r_{n-1}}{r_{n-1}}} - \lceil w_{n-2}(\xi) \rfloor^{\frac{m - r_{n-1}}{r_{n-1}}} \ ,$$

whereas $\phi(\xi)$ on the right-hand side is

$$\phi(\xi) = \left\lceil \lceil \xi_{n-1} \rfloor^{\frac{\beta_{n-1}}{r_{n-1}}} - \lceil w_{n-2} \rfloor^{\frac{\beta_{n-1}}{r_{n-1}}} \right\rfloor^{\frac{1+\tau}{\beta_{n-1}}}$$

(observe that the degree of the homogeneous sector is $1 + \tau$).

Note that $\nabla_{\xi_{n-1}} W_{n-1}(\xi)$ and $\phi(\xi)$ are of the same sign for all $\xi \neq 0$. As a consequence, there exists a positive semidefinite r-homogeneous function M defined on $\mathbb{R}^{n-1} \setminus \{0\}$ that satisfies (7b). Since the conditions of Proposition 6 are satisfied, the control υ globally asymptotically stabilises (9) whenever $-\upsilon - \delta \in [\kappa_{n-1}, \infty)_\phi$ or, equivalently, whenever $\delta \in [\kappa_{n-1} - \kappa, \infty)_{-\phi}$. $\square$

Remark 12 *The proof of Proposition 11 has an alternative interpretation. For κ not necessarily equal to κ_{n-1} we have, again by linearity of the derivative,*

$$\dot{V}(\xi) \leq -\lambda V^{\frac{m+\tau}{m}}(\xi) + \nabla_{\xi_{n-1}} V(\xi) \cdot [\delta(t, \xi) + (\kappa_{n-1} - \kappa)\phi(\xi)] \ .$$

The conditions (7b) *and $\delta \in [\kappa_{n-1} - \kappa, \infty)_{-\phi}$ imply that*

$$\nabla_{\xi_{n-1}} V(\xi) \cdot [\delta(t, \xi) + (\kappa_{n-1} - \kappa)\phi(\xi)] \leq 0 \ .$$

Thus, inequalities (14)*,* (13)*, and* (15) *are still valid in the perturbed case. As a consequence, system* (9) *is absolutely finite-time stable.*

3.2 State-Space Decomposition

Consider again (8) together with the half spaces $X_1^+ = \{x \in \mathbb{R}^n \mid x_1 \geq 0\}$, $X_1^- = \{x \in \mathbb{R}^n \mid x_1 \leq 0\}$, and their intersection $N = X_1^+ \cap X_1^-$. We will define a feedback control on $\mathbb{R}^n \setminus N$. First, partition the state as

$$x^\top = \begin{pmatrix} x_1 & \tilde{x}^\top \end{pmatrix}, \quad \tilde{x}^\top = \begin{pmatrix} x_2 & x_3 & \ldots & x_n \end{pmatrix}$$

and focus on (8a). The feedback

$$u_1^{\mathrm{a}}(x_1) = -a \operatorname{sign}(x_1) \tag{17}$$

with $a > \bar{d}_1$ steers x_1 to zero in finite time (the roman superscript 'a' will be used while rendering N attractive). Note that the convergence time is greater than or equal to $|x_1(0)|/(a + \bar{d}_1)$. Also, note that both X_1^+ and X_1^- are invariant with respect to (17).

We will now focus on (8b)-(8c). Define the matrix

$$\Theta_+ = \operatorname{diag}\left((-a)^{2-n}, (-a)^{3-n}, \dots, -a^{-1}, 1\right) \ .$$

In $\mathring{X}_1^+$, the transformations $\xi = \Theta_+ \tilde{x}$,

$$\upsilon(\xi) = u_2(x_1, \Theta_+^{-1}\xi) \ , \tag{18}$$

and $\delta(t,\xi) = d_2(t; x_1, \Theta_+^{-1}\xi)$ take (8b)-(8c) to the form (9) with the control (12). The particular form of u_2 is specified in (30b), below, while the perturbation d_2 is assumed to satisfy the sector condition given in Assumption 14, also below.

Likewise, define the matrix

$$\Theta_- = \operatorname{diag}\left(a^{2-n}, a^{3-n}, \dots, a^{-1}, 1\right) \ .$$

In $\mathring{X}_1^-$, the transformations $\xi = \Theta_- \tilde{x}$,

$$\upsilon(\xi) = u_2(x_1, \Theta_-^{-1}\xi) \ , \tag{19}$$

and $\delta(t,\xi) = d_2(t; x_1, \Theta_-^{-1}\xi)$ take the dynamics (8b)-(8c) again to (9) with the control (12).

The control (12) and the input transformations (18) and (19) suggest the feedback

$$u_2^{\mathrm{a}}(x) = -\kappa \cdot (\phi \circ h^{\mathrm{a}}(x)) \tag{20}$$

with $h^{\mathrm{a}} : \mathbb{R}^n \setminus N \to \mathbb{R}^{n-1}$ defined by

$$h^{\mathrm{a}}(x) = \begin{cases} \Theta_+ \tilde{x} & \text{if } x_1 > 0 \\ \Theta_- \tilde{x} & \text{if } x_1 < 0 \end{cases} \ . \tag{21}$$

In x-coordinates, the condition $\delta \in [\kappa_{n-1} - \kappa, \infty)_{-\phi}$ becomes the following restriction on d_2:

$$\max\ K\left[d_2 \cdot (\phi \circ h^{\mathrm{a}}) - (\kappa - \kappa_{n-1})(\phi \circ h^{\mathrm{a}})^2\right](t,x) \le 0 \ . \tag{22}$$

Let $T_S^* : \mathbb{R}^n \backslash N \to \mathbb{R}$ be the settling time for $\tilde{x}$. According to (15), it is bounded by $T_S^*(x) \le T_S(x) = T(h^{\mathrm{a}}(x))$. As a result of the previous discussion, the feedback

$$u(x) = (u_1^{\mathrm{a}}(x), u_2^{\mathrm{a}}(x))$$

drives $\tilde{x}(t, x_0)$ to zero in a time less or equal than $T_S(x_0)$ if the following condition is satisfied:

$$x(t, x_0) \notin N \quad \text{for all} \quad 0 \le t \le T_S(x_0) \ . \tag{23}$$

Mark that condition (23) is required because, once the trajectory touches N, the dynamics of $\tilde{x}$ are no longer equivalent to those of ξ and the subsequent analysis is invalid. We can now introduce the key object of the paper: a set S of initial conditions for which (23) holds. That is,

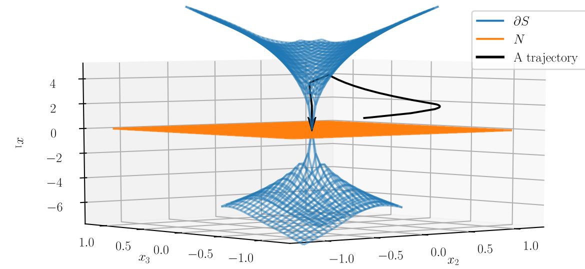

Fig. 1. Hyperplane $N = X_1^+ \cap X_1^-$ (solid orange) and boundary of the forward invariant and finite-time attractive set S (blue wireframe). A trajectory is also depicted (solid black arrow). The initial state lies outside of S, so the controller drives x away from N until it reaches S. The controller then keeps x in S and steers it towards the origin in finite time.

$$S = \{x \in \mathbb{R}^n \mid \Omega(x) \le 0\} \tag{24}$$

with

$$\Omega(x) = (a + \bar{d}_1) T_S(x) - |x_1| \ . \tag{25}$$

In other words, S is the set of all $x_0 \notin N$ for which the settling time of $\tilde{x}_0$ is guaranteed to be less than or equal to the settling time of $x_1(0)$, that is, $T_S(x_0) \le |x_1(0)|/(a + \bar{d}_1)$. Figure 1 depicts N and S for $a = 2$, $\lambda = 1$, and the parameters of Example 10.

In the following section, we will define a control law outside of S, steering trajectories towards S in finite time. The resulting closed-loop will thus have a switching surface at the boundary of S. We will show that S is attractive and forward invariant with respect to the closed-loop dynamics. This will enable us to conclude that any solution starting at $x_0 \notin N$ converges to the origin in finite time.

3.3 *Finite-Time Attractivity of S*

Let us consider system (8) with $x_0 \notin (S \cup N)$. Then, the control

$$u_1^{\mathrm{r}}(x_1) = a \operatorname{sign}(x_1) \tag{26}$$

steers x_1 away from the origin $x_1 = 0$ (the roman superscript 'r' will be used while rendering N repulsive). The control establishes the bound $|x_1(t)| \ge |x_1(0)| + (a - \bar{d}_1)t$, and also renders $\mathring{X}_1^+$ and $\mathring{X}_1^-$ invariant.

Arguing as before we see that, for $u_1 = u_1^{\mathrm{r}}$, the transformation $\xi = \Theta_- \tilde{x}$ takes (8b)-(8c) to (9) in $\mathring{X}_1^+$. In $\mathring{X}_1^-$, it is $\xi = \Theta_+ \tilde{x}$ that takes (8b)-(8c) to (9). This suggest the feedback

$$u_2^{\mathrm{r}}(x) = -\kappa \cdot (\phi \circ h^{\mathrm{r}}(x)) \tag{27}$$

with $h^{\mathrm{r}} : \mathbb{R}^n \setminus N \to \mathbb{R}^{n-1}$ defined by

$$h^{\mathrm{r}}(x) = \begin{cases} \Theta_- \tilde{x} & \text{if } x_1 > 0 \\ \Theta_+ \tilde{x} & \text{if } x_1 < 0 \end{cases} \ . \tag{28}$$

The sector condition on δ translates into the following condition on d_2:

$$\max K\left[d_2 \cdot (\phi \circ h^{\mathrm{r}}) - (\kappa - \kappa_{n-1})(\phi \circ h^{\mathrm{r}})^2\right](t,x) \leq 0 \,. \tag{29}$$

We can now establish the attractiveness of S.

Lemma 13 *Consider system (8) with controls (26) and (27), and with perturbations satisfying Assumption 7 and condition (29). For any $x_0 \notin (S \cup N)$, the solutions $x(t,x_0)$ converge to S in finite time.*

PROOF. The controls (26) and (27) ensure that, for any $\xi_0 = h^{\mathrm{r}}(x_0)$, the solution $\xi(t,\xi_0)$ is bounded for all $t \geq 0$. Hence, $\tilde{x}(t,x_0)$ remains bounded and, by continuity on $\mathbb{R}^n \setminus N$, $T_S(x(t,x_0))$ is also bounded for all $t \geq 0$. On the other hand, $|x_1(t,x_0)|$ is strictly increasing. Thus, the equality $(a+\bar{d}_1)T_S(x(t,x_0)) = |x_1(t,x_0)|$ must hold for some finite $t = t^\star > 0$. □

3.4 Forward Invariance of $\mathring{S}$

So far, we have shown that, for any $x_0 \notin (S \cup N)$, $u = (u_1^{\mathrm{r}}, u_2^{\mathrm{r}})$ brings the state x to S. We have also shown that, for any trajectory starting in $\mathring{S}$, $u = (u_1^{\mathrm{a}}, u_2^{\mathrm{a}})$ brings the state to the origin. This finally suggests the controls

$$u_1(x) = a \operatorname{sign}(\Omega(x) \cdot x_1) \tag{30a}$$

$$u_2(x) = -\kappa \lceil \sigma_{n-1}(h(x)) \rfloor^{\frac{1+\tau}{\beta_{n-1}}} \,, \tag{30b}$$

where $h : \mathbb{R}^n \setminus (N \cup \partial S) \to \mathbb{R}^{n-1}$ is defined by

$$h(x) = \begin{cases} h^{\mathrm{a}}(x) & \text{if } x \in \mathring{S} \\ h^{\mathrm{r}}(x) & \text{if } x \notin S \end{cases}$$

with h^{a} and h^{r} given by (21) and (28), respectively.

The following sector condition on the perturbation $d_2(t,x)$ synthesizes (22) and (29).

Assumption 14 *The perturbation d_2 in (8c) satisfies*

$$\max K\left[d_2 \cdot (\phi \circ h) - (\kappa - \kappa_{n-1})(\phi \circ h)^2\right](t,x) \leq 0 \,.$$

In general, the operator K renders Assumption 14 difficult to verify. A simpler sufficient condition is $|d_2(t,x)| \leq (\kappa - \kappa_{n-1})\bar{\phi}(\tilde{x})$ with $\bar{\phi}(\tilde{x}) = \min\{|\phi(\Theta_+\tilde{x})|, |\phi(\Theta_-\tilde{x})|\}$. Figure 2 depicts $\bar{\phi}$ with $\phi(\xi) = \lceil \xi_1 + \lceil \xi_2 \rfloor^{\frac{3}{2}} \rfloor^{\frac{1}{3}}$, $a = 2$ and $\kappa - \kappa_{n-1} = 1$. For comparison purposes, consider also the restriction $|d_2(t,x)| \leq \bar{\psi}(\tilde{x})$, resulting from a classical linear sector. Let us take, e.g., $\bar{\psi}(\tilde{x}) = |\psi(\Theta_-\tilde{x})|$ with $\psi(\xi) = \xi_1 + \xi_2$, and let us include it in the figure.

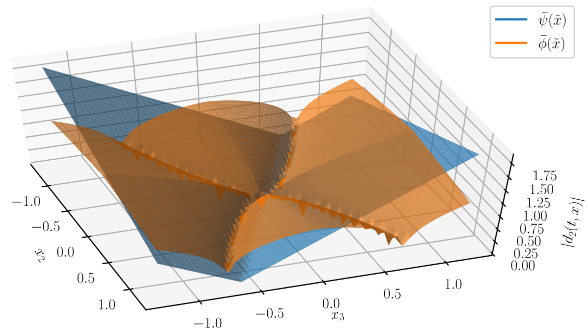

Fig. 2. Linear and nonlinear sector boundaries arising from $\bar{\psi}$ and $\bar{\phi}$, respectively.

Note that, if $|d_2|$ is known to be bounded by an r-homogeneous function of $\tilde{x}$, and of degree $0 < l_2$, then it is possible to rescale the controls and set the degree of the controlled vector-field to $l_2 - 1$ (see Remark 21).

Lemma 15 *Consider system (8) with controls (30) and perturbations satisfying Assumptions 7 and 14. The set $\mathring{S}$ is forward invariant.*

PROOF. Since $u_2(x) = u_2^{\mathrm{a}}(x)$ for $x \in \mathring{S}$, the transformation $\xi = h^{\mathrm{a}}(x)$ results in the dynamical system (9), (12). According to Remark 12, $\dot{T}(\xi) = \frac{1}{\lambda} V^{-\frac{m+\tau}{m}}(\xi)\dot{V}(\xi) < -1$. By $T_S(x) = T(h^{\mathrm{a}}(x))$, we have $\dot{T}_S(x) < -1$. Finally, from $\frac{\mathrm{d}}{\mathrm{d}t}|x_1| \geq -a - \bar{d}_1$, we can see that $\dot{\Omega}(x) < 0$ in $\mathring{S}$, as desired. □

Theorem 16 *Consider system (8) with controls (30) and perturbations satisfying Assumptions 7 and 14. The origin is almost globally finite-time attractive with basin of attraction $\mathbb{R}^n \setminus (N \cup \partial S)$.*

PROOF. Almost global finite-time attractiveness of the origin is a direct consequence of the finite-time attractiveness of S, the forward invariance of its interior and the finite-time convergence of the trajectories starting in S. □

Figure 1 also shows a trajectory of (8) with controls (30), $a = 2$, $\lambda = 1$, and the parameters of Example 10.

3.5 Stability

We have so far avoided trajectories starting on the discontinuity surfaces N and ∂S. Since all trajectories with initial conditions $x_0 \notin N \cup \partial S$ are unique, we have also eluded the discussion about multiplicity of solutions. However, establishing stability requires us to analyse such trajectories and to address the fact that solutions on $N \cup \partial S$ are non-unique [2].

Our first step is to refine Definitions 1 and 8 to handle non-uniqueness. Consider again the general differential inclusion (2) and denote by $\Phi_0(x_0)$ the set of all solutions starting at x_0.

Definition 17 *Let the origin be an equilibrium of* (1)*, i.e., let* $0 \in K[f](t,0)$*. We say that it is* almost globally finite-time weakly stable *if it satisfies the following properties:*

i) Weak Lyapunov stability: *there exists a function* $\alpha \in \mathcal{K}$ *such that* $\|x(t,x_0)\| \le \alpha(\|x_0\|)$ *for some solutions* $x(t,x_0) \in \Phi(x_0)$*.*

ii) Almost global finite-time attractivity: *there exists a subset* X_0*, dense in* $\mathbb{R}^n$*, and a locally bounded function* $T : X_0 \to \mathbb{R}_+$ *such that* $x(t,x_0) = 0$ *for all* $t \ge T(x_0)$ *and all solutions* $x(t,x_0) \in \Phi(x_0)$*,* $x_0 \in X_0$*.*

Lemma 18 *The origin is an equilibrium of* (8)*,* (30)*.*

PROOF. We will apply the operator K to (8), (30) and evaluate at $x = 0$. Regarding (8a), (8b) we have, by Assumption 7, $0 \in K[u_1 + d_1](t,0)$ and $0 \in K[u_1 \cdot 0](t,0)$. About (8c), by the continuity of u_2 at zero, $K[u_2 + d_2](t,0) = K[d_2](t,0)$. It then remains to show that $0 \in K[d_2](t,0)$. We proceed by contradiction. Suppose that $0 \notin K[d_2](t,0)$. Either $\liminf_{x\to 0} d_2(t,x) > 0$ or $\limsup_{x\to 0} d_2(t,x) < 0$ for some t. Consider the former case. For every $\varepsilon > 0$ sufficiently small, there exists $L = L(\varepsilon) > 0$ such that $d_2(t,x) \ge L$ for $x \in B(0,\varepsilon)$. By the continuity of $\phi \circ h$ at zero, we can always find $x \in B(0,\varepsilon)$ such that $\phi(h(x)) > 0$ is small enough to violate Assumption 14. The latter case can be rejected similarly. □

Lemma 19 *Consider system* (8) *with controls* (30) *and perturbations satisfying Assumptions 7 and 14. There exists a class* $\mathcal{K}$ *function* α *such that*

$$\|x(t,x_0)\| = \alpha(\|x_0\|) \; , \quad x_0 \notin N \cup \partial S \; , \qquad (31)$$

for all $t \ge 0$*.*

PROOF. Inequality (13) and the fact that $|x_1(t,x_0)|$ is non-increasing on $\mathring{S}$ imply that $\|x(t,x_0)\| \le \alpha_S(\|x_0\|)$ for $x_0 \in \mathring{S}$, for some $\alpha_S \in \mathcal{K}$, and all $t \ge 0$.

Inequality (13) also implies that

$$\|\tilde{x}(t,x_0)\| \le \tilde{\eta}(\|x_0\|) \; , \quad x_0 \notin (S \cup N) \; , \qquad (32)$$

for some $\tilde{\eta} \in \mathcal{K}$ and all $t \in [0,t^\star]$. For the same time interval and initial condition, we have the estimate $|x_1(t,x_0)| \le (a + \bar{d}_1) T_S(x(t,x_0))$. By the homogeneity of T and the piecewise linearity of h we have $|x_1(t,x_0)| \le \rho(\|\tilde{x}(t,x_0)\|)$ for some $\rho \in \mathcal{K}$ (cf. (3)). This and the bound (32) give

$$|x_1(t,x_0)| \le \eta_1(\|x_0\|) \qquad (33)$$

with $\eta_1 \in \mathcal{K}$. From (32) and (33) we conclude that $\|x(t,x_0)\| \le \eta(\|x_0\|)$ for $x_0 \notin (S \cup N)$, where $\eta(s) = \eta_1(s) + \tilde{\eta}(s)$.

The positive invariance and finite-time attractiveness of S imply (31) with $\alpha = \alpha_s \circ \eta$. □

We now turn our attention to trajectories starting on N. Since $N \cap S = \{0\}$, we have $u_1(x) = u_1^{\mathrm{r}}(x)$ for $x \in N \setminus \{0\}$, so we are interested in the initial-value problem

$$\dot{x}_1 = u_1^{\mathrm{r}}(x_1) + d_1(t,x) \; , \quad x_0 \in N \setminus \{0\} \; . \qquad (34)$$

Assumption 7 and the condition $a > \bar{d}_1$ imply that $0 \in K[u_1^{\mathrm{r}} + d_1](t,0)$ for $x \in N$, so the equilibrium $x_1(t,x_0) \equiv 0$ is a solution of (34), in which case $x(t,x_0)$ remains on N. Note, however, that the equilibrium is unstable, and that *uniqueness of solutions does not hold.* In other words, another trajectory starting on $N \setminus \{0\}$ may leave N at any time. By Lemma 19, a trajectory that leaves N at $t = 0$ satisfies $\|x(t,x_0)\| = \alpha(\|x_0\|)$, $x_0 \in \mathbb{R}^n$. A similar argument holds for ∂S. The forward invariance of $\mathring{S}$ implies that ∂S is repelling on one side, the one facing $\mathring{S}$. By the attractiveness of S, we know there is a subset $B \subset \partial S$ which is attractive on the other side, the one facing $\mathbb{R}^n \setminus S$. Thus, solutions necessarily pass from one side of B to the other and no sliding motions occur. There may be sliding motions on $\bar{B} = \partial S \setminus B$ but, since the switching surface is repulsive, solutions starting there may leave at any time. This establishes the following.

Theorem 20 *Consider system* (8) *with controls* (30) *and perturbations satisfying Assumptions 7 and 14. The origin is almost globally finite-time weakly stable.*

Mark that weak stability is sufficient in most practical scenarios, since the set $(N \cup \bar{B}) \setminus \{0\}$ is unstable and trajectories will not stay there in a physical application [25, p. 52].

The nominal system (8) with controls (30) is homogeneous of degree τ with the weights $\bar{r} = (-\tau \; r_1 \; \cdots \; r_{n-1})$, where the degree can be freely chosen, as long as it satisfies $-1 < \tau < 0$. The linearity of the nominal system with respect to (u_1, u_2) can be used to overcome the restriction and to set any desired degree.

[2] It is worth mentioning that the stability analysis in [23] is incomplete, as it disregards what happens outside S.

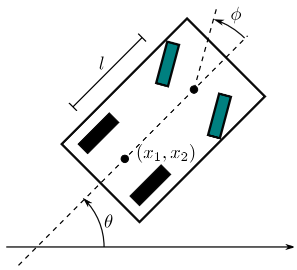

Fig. 3. Kinematic model of a car. The car configuration is parameterised by the Cartesian coordinates of the centre of the rear axle, the angle of the main body with respect to the horizontal, and the steering angle with respect to the main body. The control inputs are the linear velocity and the steering angular velocity.

Remark 21 ([23]) *For any $p \in \mathbb{N}$, the substitutions $u_1(x) \mapsto u_1(x) \cdot \|x\|_{\bar{r},p}^{\rho-\tau}$ and $u_2(x) \mapsto u_2(x) \cdot \|x\|_{\bar{r},p}^{\rho-\tau}$ render the closed-loop system $\bar{r}$-homogeneous of degree ρ.*

The new homogeneity degree, ρ, can in turn be used to modulate the convergence type (finite-time or exponential) [23], the sensitivity to noise [37], or to set the degree of the homogeneous sector for which the perturbations do not destabilise the system.

3.6 Case study: The Kinematic Car Model Revisited

We briefly recall one of the kinematic models described in [1]. Consider a car whose rear wheels are aligned with the main body and whose front wheels are allowed to rotate about the vertical axis. The distance between the front and rear axles is denoted by l (Figure 3). The car configuration is parameterised by the Cartesian coordinates of the centre of the rear axle (x_1 and x_2), the angle of the main body with respect to the horizontal (θ), and the steering angle with respect to the main body (ϕ). The control inputs are the speed of the car in the direction given by θ (v_l) and the steering velocity (ω).

The kinematic model is compatible with two nonholonomic constraints: the rear and front wheels cannot move in a direction parallel to their respective axes of rotation (the wheels do not skid). The change of coordinates $x_3 = \tan\theta$ and $x_4 = \frac{1}{l}\sec^3\theta\tan\phi$, together with the input transformations $v_l\cos\theta = u_1$ and $\omega = -\frac{3}{l}\sin^2\phi\sec\theta\tan\theta u_1 + l\cos^3\theta\cos^2\phi u_2$ bring the model to the form (9) with $n = 4$.

For $\tau = -1/2$ we have $r_1 = 2$, $r_2 = 1.5$ and $r_3 = 1$. We choose $a = 2$, $\beta_1 = \beta_2 = \beta_3 = r_1$, $\gamma_1 = \gamma_2 = 1$, and $m = 7/2$. The gains $\kappa_1 = 1$ and $\kappa_2 = 2$ yield $\sigma_3(\xi) = 4\xi_1 + 4\lceil\xi_2\rfloor^{\frac{4}{3}} + \lceil\xi_3\rfloor^2$. The controls (30) with $\kappa = \kappa_3 = 45$ render the nominal system almost globally finite-time stable. Figure 4 shows a simulation of the nominal response. The response is referred back to the original coordinates with $l = 1.5$ m (solid black).

According to Theorem 20, almost global finite-time stability is maintained in the presence of uncertainties d_1 and d_2 satisfying Assumptions 7 and 14. Assumption 7 is satisfied, e.g., by $d_1(t,x) = x_2/(x_2^2+1) + \sin(4t)/2$ with $\bar{d}_1 = 3/2 < a$. Let us compute $\sigma_3(h(x)) = x_2 + \sqrt[3]{4}\,\text{sign}(\Omega(x)\cdot x_1)\lceil x_3\rfloor^{\frac{4}{3}} + \lceil x_4\rfloor^2$. Assumption 14 is satisfied, e.g., by $d_2(t,x) = -\big(x_2 + \frac{1}{\sqrt[3]{2}}u_1(x)\lceil x_3\rfloor^{\frac{4}{3}} + \lceil x_4\rfloor^2\big)^3 \cdot (1+\cos(t))$. Figure 4 shows a simulation of the perturbed closed-loop system (dashed blue). In spite of the uncertainty, the state converges to the origin in finite-time.

When the car is subject to lateral forces, some of the wheels may loose traction, violate the constraints, and move parallel to their axes of rotation. The situation in which the front wheels loose traction is referred to as *understeer*, and it is more common in vehicles with front-wheel drive. Let v_f be the speed of the front wheels in the direction of $\theta + \phi - \frac{\pi}{2}$. When the constraints are in place, we have $v_f = 0$, but when the lateral forces are large enough, we have $v_f \neq 0$ and the model takes the perturbed chained form

$$
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= x_3 u_1 \\
\dot{x}_3 &= x_4 u_1 + \Delta f_3(x) v_f \\
\dot{x}_4 &= u_2 + \Delta f_4 v_f
\end{aligned}
$$

with

$$
\Delta f_3(x) = -\sqrt{\frac{(1+x_3^2)^3 + l^2 x_4^2}{l^2(1+x_3^2)}}
$$

and

$$
\Delta f_4(x) = \frac{3x_3x_4}{1+x_3^2}\Delta f_3(x) \ .
$$

The presence of the non-vanishing perturbation Δf_3 destroys all hope for asymptotic stability. However, note that

$$
\begin{aligned}
\Delta f_3(\delta_\varepsilon^{\bar{r}}(x)) - \frac{1}{l} &= O(\varepsilon^{\tau+\bar{r}_3}) \ , \\
\Delta f_4(\delta_\varepsilon^{\bar{r}}(x)) &= o(\varepsilon^{\tau+\bar{r}_4}) \ ,
\end{aligned}
$$

uniformly with respect to x on the homogeneous sphere $\mathbb{S}_{\bar{r},p}$. It follows from Corollary 3, item iii), that, for v_f small enough, $x(t,x_0)$ is ultimately bounded. A path for $v_f = 0.05$ m/s is shown in Figure 4 (dash-dotted orange). Indeed, the car does not reach the origin, but $x(t,x_0)$ is ultimately bounded.

Suppose now that v_f is proportional to the centripetal acceleration,

$$
v_f = v_f(x) = \alpha_f \frac{x_4}{\sqrt{1+x_3^2}} u_1(x) \ , \quad \alpha_f > 0 \ .
$$

Note that $v_f(\delta_\varepsilon^{\bar{r}}(x)) = O(\varepsilon^{\tau+\bar{r}_3})$. By Corollary 3, item ii), the closed-loop system is asymptotically stable for α_f

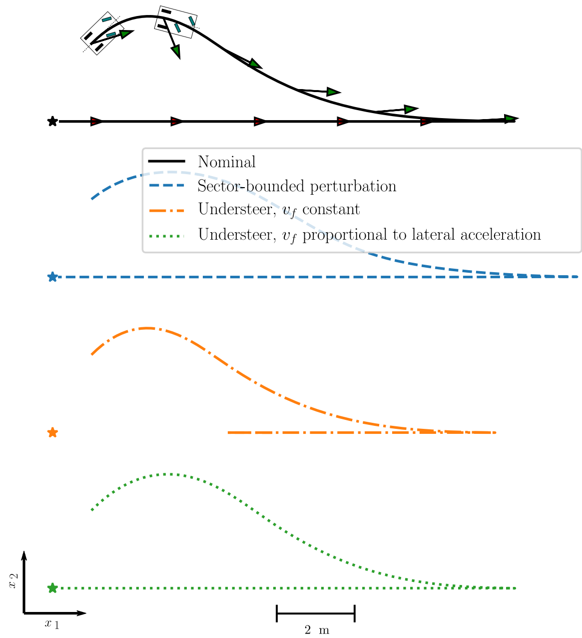

Fig. 4. Response of the closed-loop kinematic car. The lines trace the path described by x_1 and x_2. The nominal response is shown in solid black. At some points of the path, there are arrows that point in the direction of the front wheels ($\theta+\phi$). The arrow heads are coloured green and red when v_l is positive and negative, respectively. The responses to different perturbation scenarios are also included. In all scenarios, the origin $(x_1, x_2) = 0$ is marked by $\star$.

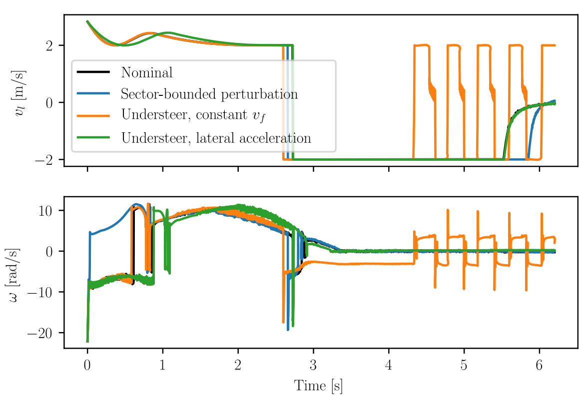

Fig. 5. Control actions for the different perturbation scenarios of the kinematic car.

small enough. Figure 4 shows a simulation for $\alpha_f = 0.2$ (dotted green). The trajectory converges to the origin.

The control actions for all scenarios are shown in Figure 5.

4 Conclusions

We considered an absolute stability problem for the nth-order nonholonomic system in chained form. The system linearisation is non-stabilisable, so the usual circle criterion cannot be applied. However, the criterion for general homogeneous systems is applicable, so it is possible to determine a class of perturbations and model uncertainties which do not destroy stability.

The distinctive benefit of the usual circle criterion is the ability to analyse and design nonlinear systems using linear tools. The new criterion effectively generalises this ability, to the degree that we can analyse and design non-homogeneous systems using homogeneous tools.

The homogeneous setting proves to be sufficiently general to overcome limitations imposed by linearisation (such as the lack of controllability) and nonholonmy (such as the lack of stabilisability by continuous feedback). Yet, the homogeneous framework is sufficiently concrete to furnish explicit control laws and Lyapunov functions. Finally, finite-time stability can be achieved simply by choosing a negative degree of homogeneity.

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