Dynamic Switching Surfaces for Output Sliding Mode Control: an $\mathcal{H}_\infty$ Approach

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Abstract

The robustness properties of sliding-mode and $\mathcal{H}_\infty$ controllers are exploited to produce a dynamic output feedback controller that is insensitive to matched perturbations and attenuates the unmatched ones. The assumptions on the plant differ from the standard assumptions of the Riccati state-space approach to $\mathcal{H}_\infty$ control. The sliding-mode controller drives the state into a reduced-order manifold for which the equivalent system does satisfy the standard assumptions and hence the standard theory can be applied. The resulting Riccati equations are of reduced order.

Key words: Sliding-mode control; Dynamic output feedback; H-infinity.

1 Introduction

$\mathcal{H}_\infty$ control is a well developed theory, at least for linear systems (see the book [6] or [14] for a tutorial). By focusing on state-space models of the plant and the controller, it is possible to characterize all proper, real-rational, internally stabilizing controllers that ensure a given upper bound γ on $\|T_{zw}\|_\infty$, the $\mathcal{H}_\infty$ norm of the transfer function that maps the perturbations w to the penalty variable z (see [7] for details on the definition). The characterization is given in terms of the solvability of two Riccati equations coupled by a simple constraint on the spectral radius of the product of the solutions [7]. Within the set of controllers achieving γ, one can extract the *central controller*, which has the same order as the plant and a nice observer-like form which is reminiscent of linear quadratic Gaussian (LQG) controllers.

The results of [7] work for plants satisfying the so-called *standard assumptions*. These assumptions can be slightly relaxed, but the resulting formulae for the Riccati equations complicates substantially [11]. On the other hand, if the Riccati equations are replaced by linear matrix inequalities (LMIs), then the standard assumptions can be almost completely removed [12]. The drawback of the LMI approach is the increased computational complexity and the lack of a simple observer interpretation for the controller.

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If one renounces to the central controller, then one obtains a set of possible controllers to choose from. This freedom has been exploited in several directions. In [13], for example, one can find a parametrization of all stabilizing controllers of order less or equal to that of the plant. In [1], $\mathcal{H}_\infty$ control is combined with $\mathcal{H}_2$ control and a mixed optimal control problem is formulated. In this paper we combine the robustness properties of $\mathcal{H}_\infty$ with sliding mode control. The motivation for doing so is best illustrated with an example.

1.1 Motivational Example

Consider the following *scalar* generalized plant

$$\dot{x} = ax + w + u \;, \quad z = \begin{bmatrix} x & u \end{bmatrix}^\top \;, \quad y = \begin{bmatrix} x & w \end{bmatrix}^\top \;,$$

where $x, w, u \in \mathbb{R}$ are the state, the perturbation and the control, respectively. The signals $y, z \in \mathbb{R}^2$ are the measured output and the penalty variable. The parameter a is considered to be positive. A controller that feeds back y into u is to be implemented, the minimization of $\|T_{zw}\|_\infty$ being the control objective. The problem just stated can be trivially solved, but we wish to illustrate

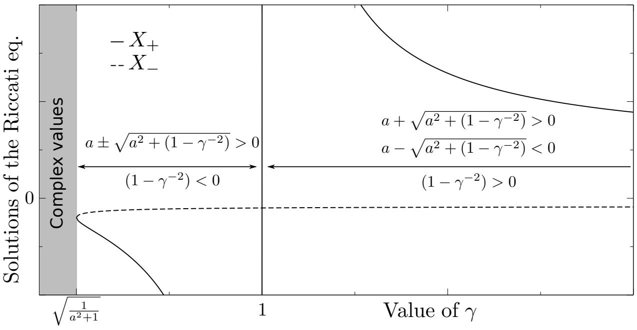

Fig. 1. Solutions of Riccati's equation. The only non negative solution (X_+) is discontinuous at the infimum value $\gamma_{\rm opt} = 1$.

on the standard method of $\mathcal{H}_\infty$ and the conditions required to apply it. This simple system complies with the assumptions required by the *full information* problem described in [7], so for a given $\gamma > 0$, there exists a controller which ensures that $\|T_{zw}\|_\infty < \gamma$ if, and only if,

$$X^2(1-\gamma^{-2}) - 2aX - 1 = 0$$

for some $X \geq 0$, the possible solutions of the quadratic equation being $X = (a \pm \sqrt{a^2 + (1-\gamma^{-2})})/(1-\gamma^{-2})$.

Consider now the problem of finding the optimal value $\gamma_{\rm opt}$. For X to be non-negative, one has $\gamma \geq 1$, which establishes the lower bound on γ. But notice that $\gamma_{\rm opt} = 1$ corresponds to the *infimum* value of γ, the actual minimum does not exist: $X \to \pm\infty$ as $\gamma \to 1$ (see Fig. 1). Since X is discontinuous at $\gamma_{\rm opt}$, one would expect a computer program performing an automated search on $\gamma_{\rm opt}$ to behave poorly. Indeed, for $a = 1$, the command `hinfsyn`[1] returns a dynamic controller along with the output

```
Test bounds:     0.0000 <  gamma  <=     0.7086
  gamma   hamx_eig  xinf_eig  hamy_eig  yinf_eig  nrho_xy   p/f
   0.709   9.2e-02 -2.7e-07   1.4e+00   5.9e-07   0.0000    p
```

The minimal eigenvalue of the solution of one of the Riccati equations equals $-2.7 \cdot 10^{-7}$, which is negative and hence violates the condition that guaranties the upper bound on the $\mathcal{H}_\infty$ norm. Nevertheless, the obtained minimal eigenvalue is very close to zero, within the bounds of a reasonable tolerance to numerical error, so the letter `p` on the far right of the output indicates that the numerical algorithm considers the test as *passed*. However, because of the discontinuity, the estimated $\hat{\gamma}_{\rm opt} = 0.7086$ presents an error of almost 30%. Moreover, the closed-loop is unstable. A more thorough discussion on the numerical aspect of $\mathcal{H}_\infty$ optimization can be found in [9,10].

Suppose now that $\bar{w}$ is a known bound for $|w|$ ($|w| < \bar{w}$) and let us take a typical sliding-mode controller

$$u = -ax - \bar{w}\,\text{sign}(x)\ .$$

[1] Robust Control Toolbox, Matlab R2009b

The dynamics of the closed-loop are $\dot{x} = w - \bar{w}\,\text{sign}(x)$, where the solutions are understood in the sense of Filippov. Clearly, x goes to zero in a finite period of time and remains there for all future time. For the purposes of analysis, when $x = 0$, an equivalent control $u_{\rm eq}$ [16] satisfying $\dot{x} = ax + w + u_{\rm eq} = 0$ is to be found. In our case $u_{\rm eq} = -w$, so $\|z\| = \|w\|$ (recall that $x = 0$), which implies that $\|z\|_2 = \|w\|_2$. ($\|\cdot\|$ denotes the Euclidean norm and $\|\cdot\|_2$ denotes the $\mathcal{L}_2$ norm). This means that the input-output gain equals $\gamma_{\rm opt}$ (see [17] for a generalization of $\mathcal{H}_\infty$ in a nonlinear setting). Thus, for this particular problem, in which we incorporate information on the bounds of the disturbances, SMC is better suited.

Sliding mode control (SMC) is a powerful and robust technique, with other interesting features such as order reduction of the dynamic equations when the system is in the sliding mode. The motivation for applying SMC also arises naturally with the use of 'on-off' low cost actuators typical in, e.g., hydraulic or pneumatic systems. Nevertheless, SMC alone has some disadvantages. One is the sensitivity to unmatched disturbances, which is usually overcome by assuming that disturbances are only of the matched type. Another disadvantage is the requirement of full state feedback, where a straightforward estimation in place of the states results in detriment of the robust properties of the controller. The problem of output feedback using SMC has been addressed in [18,5], for example. As an example of unmatched disturbance attenuation see [2].

Contribution. We construct a dynamic surface where an $\mathcal{H}_\infty$ *reduced-order* observer is used to estimate the state. We show that under reasonable assumptions the robustness properties of the SMC controller are maintained, and that when combining SMC and $\mathcal{H}_\infty$ techniques, it is possible to achieve several goals at the same time: (dynamic) output feedback, complete matched disturbance compensation and unmatched disturbance attenuation.

We take the usual approach of transforming the system into a normal (or regular) form. However, since we are considering the problem of output feedback, special care has to be taken so that not only stabilizability but also detectability is preserved by the reduced-order system. We propose a transformation that not only renders the reduced-order system stabilizable, but also detectable with respect to a certain output. Such output is not directly available, but can be recovered implicitly by changing the structure of the reduced-order $\mathcal{H}_\infty$ controller.

2 Output Regular Forms

Consider a linear time-invariant plant

$$\Sigma_{\rm p}: \quad \dot{x} = Ax + B_{\rm w}w + Bu\ , \quad y = Cx\ , \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state of the plant, $w \in \mathbb{R}^{m_1}$ is an exogenous signal containing perturbations or model uncertainties. The signal $u \in \mathbb{R}^{m_2}$ is the control and $y \in \mathbb{R}^{m_2}$ is the measured output. The matrices B and C are full-rank.

The standard regular form [16] works with systems for which the whole state can be measured. In the *output* feedback control case, a similar procedure can be carried out, but special care has to be taken so that the resulting subsystems are not only stabilizable, but also detectable [8,3]. We shall show in the following sections that the design of the sliding surface becomes simpler if the plant is in the form

$$\Sigma_{\rm pr}: \begin{cases} \dot{x}_1 \;=\; A_{11}x_1 + A_{12}x_2 + B_{11}w \\ \dot{x}_2 \;=\; A_{21}x_1 + A_{22}x_2 + B_{21}w + u \\ \;\;y \;=\; x_2 \end{cases} . \qquad (2)$$

In order to be able to transform a given system $\Sigma_{\rm p}$ into $\Sigma_{\rm pr}$ we will need the following assumption.

Assumption 1 $\Sigma_{\rm p}$ *satisfies: (i)* (A, B) *is stabilizable and* (A, C) *detectable; (ii)* $\operatorname{rank}(CB) = m_2$.

Assumption (i) is necessary and sufficient for $\Sigma_{\rm p}$ to be internally stabilizable [2] . An interpretation of (ii) is that, for the sliding mode controller to be effective, we need to measure all the state components that are matched by the controller (i.e., that belong to the span of B). This assumption can be found elsewhere in the literature [8, Ch. 5], [3].

Lemma 2 *Consider a plant* $\Sigma_{\rm p}$ *as in (1) and satisfying Assumption 1. Define the (invertible) transformation on the output* $\bar{y} = (CB)^{-1}y$*, and construct the corresponding plant*

$$\bar{\Sigma}_{\rm p}: \begin{cases} \dot{x} \;=\; Ax + B_{\rm w}w + Bu \\ \bar{y} \;=\; (CB)^{-1}Cx \end{cases} .$$

The following holds:

(I) There exists an equivalence transformation P *for which the representation of* $\bar{\Sigma}_{\rm p}$ *is in the output regular form (2), with some* B_{11} *and* B_{21} *of appropriate dimensions.*

(II) The resulting pairs (A_{11}, A_{12}) *and* (A_{11}, A_{21}) *are stabilizable and detectable, respectively.*

In other words, given a $\Sigma_{\rm p}$ satisfying Assumption 1, it can always be put in output regular form (up to the transformation $Q = (CB)^{-1}$ on the output). Without further loss of generality, we can restrict our attention to systems in output regular form.

Before stating the proof, let us define $\mathbb{C}_+$ as the closed right half complex plane. Also, let us establish some facts about pseudo-inverses. Given a matrix M, we use M^+ to denote the Moore-Penrose pseudo-inverse [15]. We denote by $M^\perp$ any matrix with a maximum number of independent columns which are orthogonal to those of M, that is, such that $M^\top M^\perp = 0$. It can be shown that $I = MM^+ + M^\perp {M^\perp}^+$, where ${M^\perp}^+$ is the Moore-Penrose pseudo-inverse of $M^\perp$ [4].

PROOF. To prove statement (I) of the lemma, consider the equivalence transformation $x' = Px$, with

$$P = \begin{bmatrix} I_{n-m_2} & 0 \\ G & I_{m_2} \end{bmatrix} \begin{bmatrix} {B^\perp}^+ \\ B^+ \end{bmatrix} = \begin{bmatrix} {B^\perp}^+ \\ G{B^\perp}^+ + B^+ \end{bmatrix} ,$$

where G is an $m_2 \times (n - m_2)$ matrix to be defined later. For any G, the inverse of P is given by

$$P^{-1} = \begin{bmatrix} B^\perp & B \end{bmatrix} \begin{bmatrix} I_{n-m_2} & 0 \\ -G & I_{m_2} \end{bmatrix} = \begin{bmatrix} B^\perp - BG & B \end{bmatrix} .$$

By directly computing

$$PB = \begin{bmatrix} 0 \\ I \end{bmatrix} , \quad PB_{\rm w} = \begin{bmatrix} {B^\perp}^+ B_{\rm w} \\ (G{B^\perp}^+ + B^+)B_{\rm w} \end{bmatrix}$$

and $CP^{-1} = \begin{bmatrix} C(B^\perp - BG) & CB \end{bmatrix}$, one can readily verify that, in x'-coordinates, $\Sigma_{\rm p}$ has the representation

$$\Sigma_{\rm p}: \begin{cases} \dot{x}'_1 \;=\; A_{11}x'_1 + A_{12}x'_2 + B_{11}w \\ \dot{x}'_2 \;=\; A_{21}x'_1 + A_{22}x'_2 + B_{21}w + u \\ \;\;y \;=\; \begin{bmatrix} C(B^\perp - BG) & CB \end{bmatrix} x' \end{cases} \qquad (3)$$

with

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} := PAP^{-1} , \quad B_{11} := {B^\perp}^+ B_{\rm w}$$

and $B_{21} := (G{B^\perp}^+ + B^+)B_{\rm w}$.

[2] We say that $\Sigma_{\rm p}$ is internally stable if $w \equiv 0$ implies that the states of the plant and the controller go to zero.

Applying the output transformation $\bar{y} = (CB)^{-1}y$ to the plant described by (3) gives

$$\bar{y} = \Big[(CB)^{-1}CB^{\perp} - G \;\; I_{m_2}\Big]\, x' \, . \tag{4}$$

Setting $G = (CB)^{-1}CB^{\perp}$ in (4) yields the desired state-space representation

$$\bar{\Sigma}_{\rm p} : \begin{cases} \dot{x}_1' &= A_{11}x_1' + A_{12}x_2' + B_{11}w \\ \dot{x}_2' &= A_{21}x_1' + A_{22}x_2' + B_{21}w + u \\ \bar{y} &= x_2' \end{cases} .$$

Note that B_{21} simplifies to $B_{21} = (CB)^{-1}CB_{\rm w}$.

To see why (II) holds, recall that a linear system is stabilizable if, and only if, $\operatorname{rank}\big[\,\lambda I - A \big| B\,\big] = n$ for all $\lambda \in \mathbb{C}_+$ (Popov-Belevitch-Hautus test). Since stabilizability is invariant under equivalence transformations, we have that

$$\operatorname{rank}\left[\begin{array}{cc|c} \lambda I_{n-m_2} - A_{11} & -A_{22} & 0 \\ -A_{21} & \lambda I_{m_2} - A_{11} & I_{m_2} \end{array}\right] = n \tag{5}$$

for all $\lambda \in \mathbb{C}_+$. The presence of I_{m_2} in the lower right corner of (5) implies that, for all λ, the last m_2 rows are independent of each other. The presence of 0 above I_{m_2} means that these columns are also independent from the rest. In other words,

$$\operatorname{rank}\Big[\,\lambda I_{n-m_2} - A_{11} \Big| -A_{22}\,\Big] = n - m_2$$

for all $\lambda \in \mathbb{C}_+$, which is equivalent to the stabilizability of (A_{11}, A_{22}). Detectability follows *mutatis mutandis.*

3 Dynamic Sliding Surfaces with an $\mathcal{H}_\infty$ Bound

Consider a plant $\Sigma_{\rm pr}$ given in the output regular form (2). This particular form makes possible the use of x_2 as a virtual control $u_{\rm v}$ for the dynamics of x_1 [16], that is,

$$\dot{x}_1 = A_{11}x_1 + B_{11}w + B_2u_{\rm v} \, , \tag{6}$$

with $B_2 := A_{12}$ and $u_{\rm v} := x_2$. Define the virtual output

$$y_{\rm v} := \dot{y} - A_{22}y - u \tag{7}$$

(later on, we will eliminate the need for $\dot{y}$). According to the third and second rows in (2), we can also write $y_{\rm v}$ as

$$y_{\rm v} = C_2x_1 + D_{21}w \tag{8}$$

with $C_2 := A_{21}$ and $D_{21} := B_{21}$ (recall that, according to Lemma 2, the pair (A_{11}, C_2) is detectable). Define the weighting function or penalty variable $z \in \mathbb{R}^{p_1}$ as

$$z = C_1x_1 + D_{12}x_2 \, , \tag{9}$$

with C_1 and D_{12} of appropriate dimensions. By binding together (6), (9) and (8) we obtain the *reduced-order* generalized plant $\Sigma_{\rm p1}$.

We now consider the sub-problem of designing a virtual controller $\Sigma_{\rm k1}$ that we will use to construct the sliding surface. We design a sliding surface along which the $\mathcal{H}_\infty$ norm of T_{zw} is bounded by some value $\gamma > 0$. The idea is to apply the standard state-space $\mathcal{H}_\infty$ theory to the reduced-order system, the main difficulty now being the fulfillment of the *standard assumptions* (which guarantee internal stability [7]) and the retrieval of the virtual output $y_{\rm v}$.

Lemma 3 *Suppose that Assumption 1 holds together with*

$$\operatorname{span}\{B, AB\} \subseteq \operatorname{span} B_{\rm w} \, . \tag{10}$$

Consider the reduced-order plant $\Sigma_{\rm p1}$ given by (6), (9) and (8). Then:

(I) (A_{11}, B_{11}) is stabilizable.
(II) There exists a matrix $J \in \mathbb{R}^{m_1 \times m_2}$ such that $J^\top \Big[B_{11}^\top \;\; D_{21}^\top\Big] = \Big[0 \;\; I_{m_2}\Big]$.

Moreover, if the matrices C_1 and D_{12} satisfy

$$D_{12}^\top \Big[C_1 \;\; D_{12}\Big] = \Big[0 \;\; \beta I_{m_2}\Big] \tag{11}$$

for some $\beta \neq 0$, then internal stability of the closed-loop formed by $\Sigma_{\rm p1}$ and an arbitrary controller $\Sigma_{\rm k1}$ is equivalent to input–output stability.

Typically, C_1 and D_{12} are design parameters, so in principle it is not difficult to fulfill (11). If the inclusion (10) is not satisfied for the original problem, it can always be enforced by adding the necessary columns to $B_{\rm w}$, i.e., by considering a larger class of perturbations — The result, though, is a more conservative controller with increased γ — The proof of the lemma, which we omit because of space constraints, follows along the same lines as [7, Lemma 16].

The following theorem shows how to recover y_v and use it to construct the dynamic sliding surface.

Theorem 4 *Consider a plant $\Sigma_{\rm pr}$ in regular form and satisfying Assumption 1,* (10) *and* (11). *Suppose that there exists positive semi-definite matrices X_∞ and Y_∞ such that*

(i) $A_{11}^\top X_\infty + X_\infty A_{11} + X_\infty(\gamma^{-2}B_{11}B_{11}^\top - B_2B_2^\top)X_\infty = -C_1^\top C_1$.

(ii) $A_{11}Y_\infty + Y_\infty A_{11}^\top + Y_\infty(\gamma^{-2}C_1^\top C_1 - C_2^\top C_2)Y_\infty = -B_{11}B_{11}^\top$.
(iii) $\rho(X_\infty Y_\infty) < \gamma^2$.

Let $\xi \in \mathbb{R}^{n-m}$ be the state of the controller and update it according to the dynamics

$$\dot{\xi} = \hat{A}_\infty \xi + (Z_\infty L_\infty A_{22} - A_\infty Z_\infty L_\infty)y + Z_\infty L_\infty u \ , \quad (12)$$

where

$$\hat{A}_\infty := A_{11} + \gamma^{-2}B_{11}B_{11}^\top X_\infty + B_2F_\infty + Z_\infty L_\infty C_2 \ ,$$
$$F_\infty := -B_2^\top X_\infty \ , \quad L_\infty := -Y_\infty C_2^\top \quad \text{and}$$
$$Z_\infty := (I_{n-m_2} - \gamma^{-2}Y_\infty X_\infty)^{-1} \ .$$

Construct the dynamic sliding surface

$$\mathcal{S} = \{(\xi, y) \mid s(\xi, y) = 0\} \ ,$$

where the switching variable s is given by

$$s(\xi, y) = F_\infty \xi - (F_\infty Z_\infty L_\infty + I_{m_2})y \ . \quad (13)$$

Then, the system trajectories along S satisfy the bound $\|z\|_2 < \gamma\|w\|_2$ and the implication $w \equiv 0 \Rightarrow \lim_{t\to\infty}(x, \xi) = 0$.

PROOF. Define the auxiliary variable

$$\hat{x}_1 = \xi - Z_\infty L_\infty y \quad (14)$$

and use (12), (14) and (7) to obtain the dynamics

$$\dot{\hat{x}}_1 = \hat{A}_\infty \hat{x}_1 - Z_\infty L_\infty y_\mathrm{v} \ . \quad (15)$$

By the definition of the sliding surface, the trajectories along it satisfy the constraint

$$y = F_\infty (\xi - Z_\infty L_\infty y) = F_\infty \hat{x}_1 \ , \quad (16)$$

but $y = x_2 = u_\mathrm{v}$, so $u_\mathrm{v} = F_\infty \hat{x}_1$. This equation, together with (15), describes the central controller achieving $\|T_{zw}\|_\infty < \gamma$ for the reduced order system $\Sigma_{\mathrm{p}1}$ (see Theorem 3 in [7]).

From Lemma 3, we have that $w \equiv 0 \Rightarrow \lim_{t\to\infty}(x_1, \hat{x}_1) = 0$. From (16) it is clear that $y = x_2 \to 0$ as $\hat{x}_1 \to 0$, so the whole state $x \to 0$ as $t \to \infty$. From (14) it is also clear that $\xi \to 0$ as $(y, \hat{x}_1) \to 0$. In conclusion, we have that $w \equiv 0 \Rightarrow \lim_{t\to\infty}(x, \xi) = 0$ and internal stability is established.

4 Enforcing the Sliding Modes

To actually enforce the sliding modes, we will need the following.

Assumption 5 *There is a known positive constant $\bar{w}$ such that $\|w\| \leq \bar{w}$ for all t. The initial value of the state of the plant satisfies $\|x_1(0)\| \leq \bar{x}_1$ for some known $\bar{x}_1$.*

Let us take $V(s) = \frac{1}{2}\|s\|^2$ as a Lyapunov function to analyse the behavior of s. The derivative of s along time is, according to (13) and (12),

$$\dot{s} = \eta(\xi, y) - (F_\infty Z_\infty L_\infty + I)y_\mathrm{v} - u \quad (17)$$

with $\eta(\xi, y) := F_\infty \left(\hat{A}_\infty \xi - Z_\infty L_\infty y\right) - A_{22}y$. The proposed control is then $u(\xi, y) = \eta(\xi, y) + M(\xi, y)\frac{s}{\|s\|}$, where $M(\xi, y)$ is a positive scalar function satisfying

$$M(\xi, y) - \|(F_\infty Z_\infty L_\infty + I)(A_{21}x_1 + B_{21}w)\| > \delta \quad (18)$$

for some positive constant δ (because of Assumption 5, such a δ always exists). The time derivative of the Lyapunov function satisfies $\dot{V} \leq -\delta\|s\| = -\delta\sqrt{2}\sqrt{V}$, which shows that s converges to zero in finite time.

5 Numerical example

Consider a plant (1) parametrized by the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix} \ , \quad B_\mathrm{w} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ ,$$

and $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. This example is motivated by the scalar example given in the introduction: it consists of an unstable plant and a perturbation with a component (w_1) that can be directly compensated by the control.

Let us define the penalty variable $(z_1, z_2) = (x, u)$. The algorithm `hinfsyn` produces a full-order controller with an $\mathcal{H}_\infty$ gain equal to 7.422. The plant is already in output regular form and $\operatorname{span}\{B, AB\} \subseteq \operatorname{span} B_\mathrm{w}$, so let us simply partition the state as $\bar{x}_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$ and $\bar{x}_2 = x_3$ and consider the reduced-order system

$$\dot{\bar{x}}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \bar{x}_1 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_\mathrm{v}$$
$$y_\mathrm{v} = \begin{bmatrix} -2 & 1 \end{bmatrix} \bar{x}_1 + \begin{bmatrix} 1 & 0 \end{bmatrix} w \ .$$

The algorithm `hinfsys`, when invoked with the output $(\bar{z}_1, \bar{z}_2) = (\bar{x}_1, 4u_\mathrm{v})$, gives a controller with an $\mathcal{H}_\infty$ gain

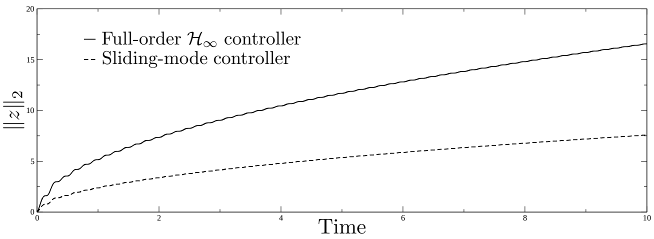

Fig. 2. The $\mathcal{L}_2$ norm of $z = (z_1, z_2) = (x, u)$. For the set of inputs $w_1 = \sin(6\pi t)$ and $w_2 = 0.5\sin(10\pi t)$, the sliding–mode controller outperforms the $\mathcal{H}_\infty$ controller.

equal to 6.7403. Based on this lower order controller, we construct a sliding-mode controller as in Theorem 4. For comparison purposes, we show (Fig. 2) the evolution of the penalty variable $(z_1, z_2) = (x, u)$, when the system starts with zero initial conditions and it is subject to the disturbances $w_1 = \sin(6\pi t)$ and $w_2 = 0.5\sin(10\pi t)$. The sliding-mode controller shows a better response than the full-order $\mathcal{H}_\infty$ controller.

6 Conclusions

Using a discontinuous control action, it was shown that there exists a stabilizing reduced-order controller satisfying $\|T_{zw}\|_\infty < \gamma$. The equivalent reduced-order model satisfies the standard assumptions, but interestingly, the original model does not. The resulting closed loop system enjoys the properties of sliding mode and $\mathcal{H}_\infty$ control: it is invariant under the action of matched disturbances, it is easy to implement and requires only partial state information.

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