Robust Output Regulation of Strongly Passive Linear Systems with Multivalued Maximally Monotone Controls

F. A. Miranda-Villatoro and Fernando Castaños

Abstract—**The use of multivalued controls derived from a special maximally monotone operator are studied in this paper. Starting with a strongly passive linear system (with possible parametric uncertainty and external disturbances) a multivalued control law is derived, ensuring regulation of the output to a desired value. The methodology used falls in a passivity-based control context, where we study how the multivalued control affects the dissipation equation of the closed-loop system, from which we derive its robustness properties. Finally, some numerical examples together with implementation issues are presented to support the main result.**

Index Terms—**Passivity-based control; Multivalued control; Robust control; Differential inclusions; Convex analysis; Hemivariational inequalities.**

I. Introduction

SOMETIMES it is useful to have an interpretation of the action of the controller in terms of energy exchange. Among the most important methodologies of passivity-based control (PBC) that achieve this interpretation are the so-called energy-shaping techniques. The purpose of energy shaping, as its name suggests, is to change the energy function by means of the control action in such a way that stabilization and performance objectives are satisfied. Although energy-shaping strategies have proved to be very useful yielding an easy interpretation of the controller in terms of energy exchange (see, e.g., [1], [2], [3]), robustness against external perturbations and model uncertainty is still a topic of research.

Among the most common energy-shaping techniques one finds energy-balancing passivity-based control and interconnection and damping assignment (IDA) [3]. In the energy-balancing approach, energy shaping is accomplished by writing the total energy of the closed-loop system as the sum of the energy of the open-loop system and the energy of the controller [1], [4]. It can be shown that, for mechanical systems, this approach is limited to the shaping of the potential energy alone. In the second approach, the to-be-controlled system is assumed to be port-Hamiltonian with given Hamiltonian (energy) function and interconnection and damping matrices. Energy shaping is accomplished by matching the closed-loop system to another port-Hamiltonian system with desired parameters. The question of whether or not it is possible to establish an energy balance for the controller is disregarded. For mechanical systems, this approach allows to modify both the potential and the kinetic energies. The interested reader is addressed towards [1], [5], [6], [7] for detailed accounts on passivity-based control.

The study of differential inclusions for modelling and analysis of processes in control theory is extensive (e.g. [8], [9], [10], [11]), whereas the problem of designing a multivalued control in order to achieve a desired response is less explored, except for the case of sliding-mode control [12], which takes advantage of the multivalued nature of the signum multifunction to ensure robustness of the closed-loop system.

An important family of differential inclusions (more general than those obtained by using sliding-mode techniques) are those for which its right-hand side is represented by maximally monotone operators. In the case of linear plants, the closed-loop system is sometimes called a multivalued Lur'e dynamical system, and results about existence and uniqueness of solutions have been proved in [13], [14], [15], [16]. This kind of systems are related to complementarity and projected dynamical systems [17], which makes its study important for a broad range of applications coming from different fields such as automatic control, economics, mechanics etc.

Recently, the use of maximally monotone operators for the control of systems was presented in [18], where the authors consider the design of a state feedback control law for systems of Lur'e type with multivalued right-hand side and developed a static and a dynamic control law which depends on both the system parameters and the system state.

The main contribution of this paper consists of a design procedure for a multivalued-control — where the multivalued part is represented by the subdifferential of some proper, convex and lower semicontinuous function — which achieves finite-time regulation of the desired output together with insensitivity in the face of a family of bounded and unmatched perturbations.

The proposed multivalued control strategy differs remarkably from those which are common in sliding-mode control in the sense that we obtain finite-time regulation and disturbance rejection without a discontinuous right-hand side and therefore without the necessity of solutions of the associated system in the sense of Filippov. Moreover, in this work we focus on systems with zero-relative degree (i.e., the output of the system depends explicitly on the control input) which, to the best of the authors knowledge, cannot be treated using conventional sliding-modes techniques.

This paper is organized as follows. In Section II the class

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The authors are with the Department of Automatic Control, CINVESTAV-IPN, Av. Instituto Politécnico Nacional 2508, 07360 Mexico City, Mexico (e-mail: fmiranda; fcastanos@ctrl.cinvestav.mx)

of systems that we consider is established in conjunction with the class of perturbations that will be treated. The multivalued structure of the controller is presented and well-posedness of the closed-loop system is established. In Section III we introduce the main result of this paper. Namely, robustness and finite-time convergence of the closed-loop system are demonstrated. Section IV touches the point about implementation of the multivalued control law by introducing a regularization of the multivalued map. Some examples are presented showing the properties of the closed loop. The paper ends with some conclusions and future research lines in Section V.

A. Notation and preliminaries

Throughout this paper, all vectors are column vectors, even the gradient of a scalar function that we denote by $\nabla H(x) = \frac{\partial H(x)}{\partial x}$.

A matrix $A \in \mathbb{R}^{n\times n}$ is called positive definite, denoted as $A > 0$, if $w^\top A w > 0$ for all $w \in \mathbb{R}^n \setminus \{0\}$[1]. The minimum and the maximum eigenvalues of a symmetric matrix $B \in \mathbb{R}^{n\times n}$ are denoted as $\lambda_{\min}(B)$ and $\lambda_{\max}(B)$ respectively.

A set-valued function or multifunction $\mathbf{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a map that associates with any $w \in \mathbb{R}^n$ a subset $\mathbf{F}(w) \subset \mathbb{R}^n$. The domain of $\mathbf{F}$ is given by

$$\operatorname{Dom}\mathbf{F} = \{w \in \mathbb{R}^n : \mathbf{F}(w) \neq \emptyset\} \ .$$

Related to the definition of a multifunction is the concept of its graph,

$$\operatorname{Graph}\mathbf{F} = \{(w,z) \in \mathbb{R}^n \times \mathbb{R}^n : z \in \mathbf{F}(w)\} \ .$$

The graph is used to define the concept of monotonicity in the following way: A set-valued function $\mathbf{F}$ is said to be monotone if for all $(w,z) \in \operatorname{Graph}\mathbf{F}$ and all $(w',z') \in \operatorname{Graph}\mathbf{F}$ the relation

$$\langle z - z', w - w'\rangle \geq 0$$

is preserved, where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product on $\mathbb{R}^n$. A monotone map $\mathbf{F}$ is called maximally monotone if, for every pair $(\hat{w},\hat{z}) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \operatorname{Graph}\mathbf{F}$, there exists $(w,z) \in \operatorname{Graph}\mathbf{F}$ with $\langle z-\hat{z}, w-\hat{w}\rangle < 0$, or in other words, if no enlargement of its graph is possible in $\mathbb{R}^n \times \mathbb{R}^n$ without destroying monotonicity.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous function. The effective domain of f is given by

$$\operatorname{Dom} f = \{w \in \mathbb{R}^n : f(w) < \infty\} \ .$$

We say that f is proper if its effective domain is non empty. The subdifferential $\partial f(w)$ of $f(\cdot)$ at $w \in \mathbb{R}^n$ is defined by

$$\partial f(w) = \{\zeta \in \mathbb{R}^n : f(\sigma) - f(w) \geq \langle \zeta, \sigma - w\rangle \quad \text{for all } \sigma \in \mathbb{R}^n\} \ .$$

[1]According to this definition, a matrix A is positive definite if, and only if, its symmetric part is positive definite. For convenience, most authors assume that A is already symmetric. However, for our purposes it will be more convenient not to make such assumption (see, e.g., [19]).

An important convex function is the indicator function of a convex set S, defined by

$$\psi_S(w) = \begin{cases} 0 & \text{if } w \in S \\ +\infty & \text{if } w \notin S \end{cases} .$$

It is easy to see that, when $f(\cdot)$ is equal to the indicator function of a closed convex set S, then the subdifferential coincides with the normal cone of the set S at the point $w \in S$, i.e.,

$$\partial\psi_S(w) = N_S(w) = \{\xi \in \mathbb{R}^n : 0 \geq \langle \xi, \sigma - w\rangle \quad \text{for all } \sigma \in S\} \ .$$

Note that if w is in the interior of S then $N_S(w) = \{0\}$. If $w \notin S$ then $N_S(w) = \emptyset$.

II. The output regulation problem

Consider the following affine system:

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + B_{\mathrm{u}}u_1(t) + B_{\mathrm{v}}v(t) \\ y_1(t) = Cx(t) + Du_1(t) \end{cases} , \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the system state, $u_1, y_1 \in \mathbb{R}^m$ are the port variables available for interconnection, and matrices $A, B_{\mathrm{u}}, B_{\mathrm{v}}, C, D$ are constant and of suitable dimensions. The term $v \in \mathbb{R}^m$ accounts for an uncertain exogenous input which is considered bounded in the $\mathcal{L}_\infty$ sense, i.e., $\sup_{t\in\mathbb{R}_+}\|v(t)\| < \infty$. Moreover, without loss of generality, the external signal $v(t)$ can be decomposed as the sum of a constant term v^+ and a bounded signal $\nu(t)$.

The robust output regulation problem consists in regulating the output y_1 to a desired value y_{d}, even in the presence of the external perturbation $v(t)$ and of parametric uncertainties.

Remark 1. Notice that, for $D = 0$ and $B_{\mathrm{u}} = B_{\mathrm{v}}$, the problem reduces to a standard sliding-mode control problem with matched disturbances. We depart from these standard assumptions and make the following instead.

Assumption 1. *System* (1) *is a minimal realization and is strongly passive. That is, there exists a (possibly unknown) matrix* $P = P^\top > 0$ *such that*

$$\mathcal{R} = \begin{bmatrix} PA + A^\top P & PB_{\mathrm{u}} - C^\top \\ B_{\mathrm{u}}^\top P - C & -(D + D^\top)\end{bmatrix} < 0 \ . \tag{2}$$

Consider the function $H_0(x) = \frac{1}{2}x^\top P x$ with P satisfying (2). Taking the time-derivative of H_0 along the system trajectories (with $v(t) = 0$, for all t) gives the energy balance

$$\begin{aligned}\dot{H}_0(x) &= x^\top P\dot{x} \\ &= \frac{1}{2}x^\top(A^\top P + PA)x + u_1^\top B_{\mathrm{u}}^\top P x \\ &\quad + u_1^\top(y_1 - Cx - Du_1) \\ &= \frac{1}{2}w^\top \mathcal{R} w + u_1^\top y_1 \ ,\end{aligned}$$

where $w^\top = \begin{bmatrix} x^\top & u_1^\top \end{bmatrix}$ and with $\mathcal{R}$ as in (2). Hence, H_0 is a storage function for system (1). Moreover, from the energy balance we obtain

$$\dot{H}_0(x) \leq -\frac{\lambda_{\min}(-\mathcal{R})}{2}\|x\|^2 + u_1^\top y_1 - \frac{\lambda_{\min}(-\mathcal{R})}{2}\|u_1\|^2 \ .$$

Therefore, the strong passivity assumption (2) implies input strict passivity together with a positive definite dissipation term. See [20] for more details.

Notice that (2) also implies that D is positive definite (hence non singular). Depending on the application at hand, the somewhat stringent condition on the invertibility of D can be circumvented in several ways. If the plant is passive with an output of relative degree equal to one ($D = 0$). It is possible to generate a complete family of passive outputs with relative degree equal to zero and parametrized precisely by D (see [4] for the nonlinear Hamiltonian case). Another possibility is to take the first i successive derivatives of $y = Cx$ such that $y^{(i)}$ depends explicitly on u. Then take $y^{(i)}$ as the new output and test for strong passivity.

It is worth noting that, in the linear case, the class of passive systems is equivalent to the class of port-Hamiltonian (PH) systems described in [5, Ch. 4], i.e., Σ can be written as

$$\begin{aligned} \dot{x} &= F\nabla H_0(x) + g_{\mathrm{u}} u_1 + g_{\mathrm{v}} v \\ y_1 &= h(x) + j u_1 \end{aligned}$$

with $F = AP^{-1} = J - R$, where $J = -J^\top$ and $R = R^\top \geq 0$ are the so-called interconnection and dissipation matrices, respectively, $g_{\mathrm{u}} = B_{\mathrm{u}}$, $g_{\mathrm{v}} = B_{\mathrm{v}}$, $h(x) = Cx$ and $j = D$. Along this paper we will use both representations of Σ with the purpose of expressing the related computations in the context of basic IDA [3], [5].

A. Multivalued control law

In this subsection a multivalued control law is introduced by using maximally monotone operators. It will be shown later that these are robust in the face of parametric and additive uncertainties.

Let $u_2 \in \mathbb{R}^m$ and $y_2 \in \mathbb{R}^m$ be the controller port variables. The multivalued control input is defined in terms of the graph of a multifunction $\mathbf{U} : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ by

$$(u_2, y_2) \in \operatorname{Graph} \mathbf{U} \; .$$

Remark 2. It is worth mentioning that, in the case when the multifunction $\mathbf{U}$ is monotone, the relation $(u_2, y_2) \in \operatorname{Graph} \mathbf{U}$ defines a static, incrementally passive map[2]. Furthermore, if $0 \in \mathbf{U}(0)$, then the relation between u_2 and y_2 defines a static passive map inasmuch as

$$\langle u_2, y_2 \rangle \geq 0 \text{ for all } (u_2, y_2) \in \operatorname{Graph} \mathbf{U} \; .$$

Previous lines motivate the following assumption.

Assumption 2. *The multifunction* $\mathbf{U}$ *is maximally monotone and defines a static passive relation between the input* u_2 *and the output* y_2*, i.e.,* $0 \in \mathbf{U}(0)$.

The multivalued nature of the proposed control motivates us to depart from the classical intelligent control paradigm and to instead make use of the behavioural framework proposed by Willems [22]. In this context, the plant and the controller

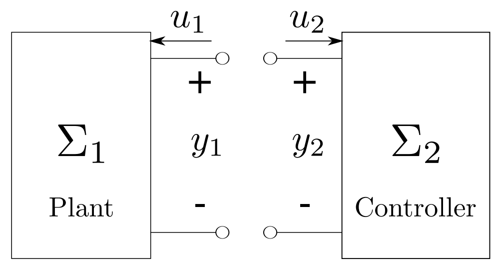

Fig. 1. Interconnection of a controller to a plant.

are interconnected using a power preserving pattern as shown in Figure 1, satisfying $y_1 = y_2 =: y$, $-u_1 = u_2 =: u$ and therefore $u_1 y_1 + u_2 y_2 = 0$.

The interconnected system (plant and controller) results in

$$\dot{x} = Ax - B_{\mathrm{u}} u + B_{\mathrm{v}} v \tag{3a}$$

$$y = Cx - Du \tag{3b}$$

$$u \in \mathbf{U}(y) \; , \tag{3c}$$

where our task is to determine $\mathbf{U}(y)$ such that y is regulated to some fixed value y_{d}, even in the presence of uncertainties in the system parameters and the external perturbation v. Note that the previous constraint rules out the trivial control $u = D^{-1}(Cx - y_{\mathrm{d}})$. In fact, even if all the system parameters and the state x were known, that control would not be admissible, since it is not passive (see Assumption 2).

It is well known that, when $\mathbf{U}(y)$ is given as the subdifferential of a proper, convex and lower semicontinuous function $\Phi(\cdot)$ (i.e., when $\mathbf{U}(y) = \partial\Phi(y)$), it is a maximally monotone operator [23, Cor. 31.5.2]. Therefore, we will focus on controls of the form

$$u(t) \in \partial\Phi(y(t)) \text{ for all } t \geq 0 \; , \tag{4}$$

with $\Phi : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ some proper, convex and lower semicontinuous function. More specifically, in Section III we will prove that, for some closed convex set S, robust regulation of the output y is obtained for the case when $\Phi(y) = (\varphi + \psi_S)(y)$, where $\varphi(\cdot)$ is proper, convex and lower semicontinuous with effective domain containing S and satisfying $0 \in \partial\varphi(0)$. The function $\psi_S(\cdot)$ is the indicator function of the set S. In other words, Φ is the restriction of φ to S.

B. Well-posedness

Before presenting the main result on the robustness of the closed-loop system (3), it is important first to establish its well-posedness. Specifically, well-posedness of the closed-loop system comprises two issues. The first question is: Is there always a control input $u \in \partial\Phi(y)$? and the second one is about uniqueness and existence of solutions of the associated differential inclusion (3).

Regarding the second issue, about a solution of the differential inclusion (3), well-posedness was proved previously in [14], [15], where the subdifferential of the conjugate function of $\Phi(y)$ together with passivity of the associated system play a crucial role.

[2] A static multivalued map $\mathbf{F}$ is incrementally passive if $\langle y - y', u - u' \rangle \geq 0$ for all $(u, y), (u', y') \in \operatorname{Graph} \mathbf{F}$. The interested reader is kindly addressed to [21] for details on dynamical incrementally passive maps.

The first issue deserves more explanation. At first we need $y(t) \in S$ for all time t, this comes from the definition of the subdifferential, i.e., $u(t) \in \partial\Phi(y(t))$ is equivalent to

$$\Phi(\sigma) - \Phi(y) \geq \langle u, \sigma - y\rangle \text{ for all } \sigma \in \mathbb{R}^m ,$$

where we have omitted the time argument. In the case of $\Phi(y) = (\varphi + \psi_S)(y)$ we have

$$\varphi(\sigma) - \varphi(y) + \psi_S(\sigma) - \psi_S(y) \geq \langle u, \sigma - y\rangle \text{ for all } \sigma \in \mathbb{R}^m , \tag{5}$$

and it is clear that if $y \notin S$, then we will have $\partial\Phi(y) = \emptyset$. Thus, we must guarantee that, no matter what the initial conditions are, it is possible to find an output $y \in S$ such that $u \in \partial\Phi(y)$ is well defined.

In the case where $\varphi \equiv 0$ and D is symmetric, well-posedness is easy to show. Since $u \in \partial\psi_S(y) = N_S(y)$, from the definition of a normal cone we have

$$0 \geq \langle u, \sigma - y\rangle \text{ for all } \sigma \in S ,$$

which in view of (3b) translates into

$$0 \geq \langle D^{-1}(Cx - y), \sigma - y\rangle = \langle Cx - y, \sigma - y\rangle_{D^{-1}}$$

for all $\sigma \in S$, with the inner product weighted by D^{-1} (recall that D^{-1} is well defined in view of Assumption 1). From [24, p. 117] we have that the above inequality is the characterization of the projection of Cx onto the set S with the induced norm $\|\cdot\|_{D^{-1}}$, i.e.,

$$y = \mathrm{Proj}_S^{D^{-1}}(Cx) = \arg\min_{\sigma\in S} \|Cx - \sigma\|_{D^{-1}} , \tag{6}$$

and the control input u transforms into

$$u = D^{-1}\left(Cx - \mathrm{Proj}_S^{D^{-1}}(Cx)\right) \in N_S(y) . \tag{7}$$

Therefore, when D is symmetric we can find an expression for the output y in terms of the projection operator $\mathrm{Proj}_S^{D^{-1}}(\cdot)$ (note that this implies $y \in S$ independently of the state x). Moreover, due to the Lipschitzian property of the projection operator [24, p. 118], substitution of u in (3) leads to a well-posed ordinary differential equation (not a differential inclusion!) with a Lipschitzian right-hand side (see [25] for a detailed development in the scalar case).

For the general case where φ is not the zero function, and removing the assumption about the symmetry of D, from (5) we have that the problem consists of finding $y \in S$ such that

$$0 \leq \langle D^{-1}y - D^{-1}Cx, \sigma - y\rangle + \Phi(\sigma) - \Phi(y) \text{ for all } \sigma \in S , \tag{8}$$

where we made use of (3b). The inequality (8) is an hemi-variational inequality[3] for which existence and uniqueness of solutions can be deduced from the invertibility of D, as the following lemma shows.

Lemma 1. *[27, Lemma 5.2.1] Suppose that $F(y)$ is continuous and strongly monotone, i.e.*

$$\langle F(y) - F(y'), y - y'\rangle \geq \alpha\|y - y'\|^2$$

for all $y, y' \in \mathbb{R}^m$ and some $\alpha > 0$ and let $\Phi : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function with effective domain $S \subset \mathbb{R}^m$. Then, for each $g \in \mathbb{R}^n$ there exists a unique solution $y^* \in S$

$$\langle F(y) - g, \sigma - y\rangle + \Phi(\sigma) - \Phi(y) \geq 0 \textit{ for all } \sigma \in S .$$

Applying the previous Lemma to our problem (8), we have that existence and uniqueness of solutions is immediate. Since D is positive definite, it is straightforward to see that D^{-1} is positive definite too. Furthermore, the linear map $y \mapsto D^{-1}y$ is strongly monotone. Indeed,

$$\begin{aligned}\langle D^{-1}(y_1 - y_2), y_1 - y_2\rangle &= (y_1 - y_2)^\top D_S^{-1}(y_1 - y_2)\\ &\geq \lambda_{\min}\left(D_S^{-1}\right)\|y_1 - y_2\|^2 ,\end{aligned}$$

where $D_S^{-1} = (D^{-1} + D^{-\top})/2$ is the symmetric part of D^{-1}. Consequently, the hemivariational inequality (8) has a unique solution for each state x. In other words: For all $x \in \mathbb{R}^n$, there always exists a unique $y \in S$ such that the control $u \in \partial\Phi(y)$ is well defined.

Remark 3. The computation of the control input which forces $y \in S$ obviously depends on the solution of the hemivariational inequality (8) and therefore it depends implicitly on the actual state of the plant x. This dependency on the state induces a partition in the phase space and leads to a hybrid system.

Remark 4. For the case when $\varphi \equiv 0$ and D is symmetric, we might be tempted to use (7) as a control input (because it is passive) but, unfortunately, it is not implementable in our setting because it depends explicitly on the system parameters and state. The role of (7) is analogous to the role of the equivalent control in sliding modes [12], in the sense that it is not implementable but it helps to determine the dynamics associated to the closed-loop system. See [25] for an example on the use of the control (7) in the scalar case and some implementation issues.

Following the steps in [25], the control that results from the solution of the hemivariational inequality (8) will act as an equivalent control, in the sense of Remark 4 and is not implementable under the assumption that the state and the plant parameters are unknown. The implemented control is described in Section IV.

III. Finite-time perfect output regulation

The main result of this paper is presented in this section. Namely, from an energy-shaping point of view, we show that the multivalued control (4) can be expressed as a basic IDA controller plus a robustifying term denoted by η, affecting directly the dissipation of the closed-loop system and yielding to the output regulation despite the presence of external and parametric disturbances.

Through the rest of this paper we will consider the interconnected system (3) with multivalued control (4) and $\Phi(\cdot) = (\varphi + \psi_S)(\cdot)$ with S some convex set. The perturbation input $v(t)$, decomposed as a constant term v^+ and a bounded unknown signal $\nu(t)$, affects the dissipation equation in the following way.

[3] The interested reader is referred to [26], [27], [28], and references therein for more information and properties about variational and hemivariational inequalities.

Let $\bar{x}$ be the equilibrium point of (3a) associated to a constant perturbation ($\nu(t) \equiv 0$) and input $u = 0$, i.e.,

$$0 = A\bar{x} + B_{\mathrm{v}} v^{+} \ , \tag{9}$$

and let H_0 be the storage function of system (3a)–(3b) (i.e., $H_0(x) = \frac{1}{2} x^\top P x$ with P satisfying (2)). We obtain

$$\begin{aligned} 0 &= F\nabla H_0(\bar{x}) + B_{\mathrm{v}} v^{+} \\ &= F\nabla H_0(\bar{x}) \pm F\nabla H_0(x) + B_{\mathrm{v}} v^{+} \\ &= -F\left(\nabla H_1(x) - \nabla H_0(x)\right) + B_{\mathrm{v}} v^{+} \end{aligned}$$

with $H_1(x) = \frac{1}{2}(x-\bar{x})^\top P(x-\bar{x})$. Now, defining $H_a(x) = H_1(x) - H_0(x)$ we have the basic IDA controller equation [4] for v^{+} as

$$F\nabla H_a(x) = B_{\mathrm{v}} v^{+} \ .$$

Thus, we have that the term v^{+} acts as an energy-shaping control changing the storage function of the uncontrolled system H_0 to H_1 and therefore changing the equilibrium of the system. The closed-loop system results in

$$\dot{x} = F\nabla H_1(x) - B_{\mathrm{u}} u + B_{\mathrm{v}} \nu \tag{10a}$$

$$y = Cx - Du \tag{10b}$$

$$u \in \partial\Phi(y) \tag{10c}$$

For the case $\nu = 0$, a control input can be designed in order to obtain the asymptotic regulation of the output y to y_{d} using an energy-shaping interpretation as follows.

Lemma 2. *For system* (10a)–(10b)*, assume that x_* is an admissible equilibrium associated to the constant control $\bar{u} = D^{-1}(Cx_* - y_{\mathrm{d}})$, i.e., x_* satisfies*

$$0 = Ax_* - B_{\mathrm{u}} D^{-1}(Cx_* - y_{\mathrm{d}}) + B_{\mathrm{v}} v^{+} \ . \tag{11}$$

Then, $\bar{u}$ achieves regulation of the output to y_{d} when $\nu = 0$. Furthermore, $\bar{u}$ is a basic IDA controller and satisfies

$$F\nabla H_b(x) = -B_{\mathrm{u}} \bar{u}$$

with $H_b(x) = H_2(x) - H_1(x)$ and

$$H_2(x) = \frac{1}{2}(x - x_*)^\top P (x - x_*) \ .$$

Proof. Let x_* be an equilibrium of system (10) satisfying (11). Then, from (9) we have that

$$0 = A(x_* - \bar{x}) - B_{\mathrm{u}} D^{-1}(Cx_* - y_{\mathrm{d}})$$

or, in terms of the storage functions H_1 and H_2,

$$0 = -F(\nabla H_2(x) - \nabla H_1(x)) - B_{\mathrm{u}} D^{-1}(Cx_* - y_{\mathrm{d}}) \ .$$

Therefore, we obtain a change in the storage function from $H_1(x)$ with minimum at $\bar{x}$ to $H_2(x)$ with minimum at x_*, which implies convergence of the state x to x_*. Also, for $u = \bar{u}$ in (10b) we have

$$y = Cx - DD^{-1}(Cx_* - y_d) = C(x - x_*) + y_d$$

and $y \to y_d$ as $x \to x_*$.

□

The control $\bar{u}$ described in Lemma 2 shapes the energy by changing the storage function. For the new storage function H_2 we have that the control input $u = \bar{u} + \eta$, where $\eta : [0, +\infty) \to \mathbb{R}^m$ and is specified below, establishes a new dissipation equation as

$$\begin{aligned} \dot{H}_2(x) &= \nabla H_2(x)^\top \dot{x} \\ &= \nabla H_2(x)^\top F \nabla H_2(x) - \nabla H_2(x)^\top B_{\mathrm{u}} \eta \\ &\quad + \nabla H_2(x)^\top B_{\mathrm{v}} \nu \\ &= \frac{1}{2}(x - x_*)^\top (A^\top P + PA)(x - x_*) \\ &\quad - (x - x_*)^\top P B_{\mathrm{u}} \eta - (y - Cx + D(\bar{u} + \eta))^\top \eta \\ &\quad + (x - x_*)^\top P B_{\mathrm{v}} \nu \\ &= \frac{1}{2} \begin{bmatrix} (x - x_*)^\top & -\eta^\top \end{bmatrix} \mathcal{R} \begin{bmatrix} x - x_* \\ -\eta \end{bmatrix} \\ &\quad - (y - y_{\mathrm{d}})^\top \eta + (x - x_*)^\top P B_{\mathrm{v}} \nu \ . \end{aligned} \tag{12}$$

In the case $\nu = 0$ we obtain the energy-balancing equation changing the output to $-(y - y_{\mathrm{d}})$.

Remark 5. The control $\bar{u}$ achieves the asymptotic regulation of the output y via a change in the storage function $H_1(x)$ but, once again, $\bar{u}$ is not implementable, as it requires perfect knowledge of the state and system parameters and would lead to a closed-loop system which is not robust.

In Section II-B it was established that, when $\Phi(y) = \psi_S(y)$ and D is symmetric, we have

$$y = \mathrm{Proj}_S^{D^{-1}}(Cx) \ .$$

This equation evidently shows the robustness property of the multivalued control law $u \in N_S(y)$, since it is not necessary to maintain the state x at a precise point. Instead, it is sufficient to maintain x in the set of points for which its projection over S is equal to y_d. This property obviously depends on the shape of the set S and, in order to achieve robust regulation, it is necessary to have $y_d \in \partial S$ and $\mathrm{int}\, N_S(y_{\mathrm{d}}) \neq \emptyset$ (see Figures 2, 3 below).

Assumption 3. *The set $S \subset \mathbb{R}^m$ is closed and convex. Moreover, $y_{\mathrm{d}} \in S$ is such that* $\mathrm{int}\, N_S(y_{\mathrm{d}}) \neq \emptyset$.

The previous argument can be extended to the more general case where $u \in \partial\Phi(y)$ with $\Phi(y) = (\varphi + \psi_S)(y)$ and φ an arbitrary proper, convex and lower semicontinuous function, i.e., we can achieve robust output regulation for a family of controls parametrized by φ.

Theorem 1 (Main result)**.** *Consider system* (10)*, suppose that Assumption 1 holds and consider the family of controls $u \in \partial\Phi(y)$, parametrized by φ as $\Phi(y) = (\varphi + \psi_S)(y)$, with φ proper, convex, lower semicontinuous and such that $0 \in \partial\varphi(0)$. S is some closed convex set specified along the proof. Then, the control* (10c) *yields the robust output regulation $y = y_{\mathrm{d}}$ in finite time whenever*

$$\langle D^{-1}(y_{\mathrm{d}} - Cx_*), y_{\mathrm{d}} \rangle < \mathcal{D}\varphi(y_{\mathrm{d}}, -y_{\mathrm{d}}) \ , \tag{13}$$

and

$$\|\nu\| \le \mathcal{B}$$

for some $\mathcal{B} > 0$, also specified along the proof. Here, $\mathcal{D}\varphi(y_0, d)$ is the directional derivative of the function φ at the *point y_0 in the direction d and x_* is the equilibrium associated to the basic IDA design (Lemma 2).*

Proof. Applying the control input $u \in \partial\Phi(y)$ automatically implies that $y \in S$ (see Subsection II-B). Then, if we want the regulation of y to y_{d}, a necessary condition is $y_{\mathrm{d}} \in S$. Consider the following convex set

$$S = \operatorname{conv}\{0, y_{\mathrm{d}}\} \ , \tag{14}$$

where the operator $\operatorname{conv}\{a, b\}$ refers to the convex hull of two points $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^m$, i.e.,

$$\operatorname{conv}\{a, b\} = \{c \in \mathbb{R}^m : c = \lambda a + (1-\lambda)b,\ \lambda \in [0,1]\} \ ,$$

and consider the half-space

$$\Omega_{\mathrm{d}} = \{x \in \mathbb{R}^n : \langle D^{-1}(y_{\mathrm{d}} - Cx), y_{\mathrm{d}}\rangle \leq \mathcal{D}\varphi(y_{\mathrm{d}}, -y_{\mathrm{d}})\} \ .$$

Our first goal is to show that the value of the output y is equal to y_{d} if, and only if, $x \in \Omega_{\mathrm{d}}$. The inclusion $x \in \Omega_{\mathrm{d}}$ implies

$$\begin{aligned}\langle D^{-1}(y_{\mathrm{d}} - Cx), y_{\mathrm{d}}\rangle &\leq \mathcal{D}\varphi(y_{\mathrm{d}}, -y_{\mathrm{d}}) \\ &= \lim_{\rho\downarrow 0} \frac{\varphi(y_{\mathrm{d}} - \rho y_{\mathrm{d}}) - \varphi(y_{\mathrm{d}})}{\rho} \\ &\leq \frac{\varphi(y_{\mathrm{d}} - \rho y_{\mathrm{d}}) - \varphi(y_{\mathrm{d}})}{\rho} \text{ for all } \rho > 0 \ ,\end{aligned}$$

where in the last inequality we have used the fact that, for a convex function, the map $t \mapsto \frac{\varphi(v+td)-\varphi(v)}{t}$ is increasing [29, Prop. 17.2]. From the last inequality it is possible to obtain a new one in terms of the elements of S. First, we restrict the values of ρ to the interval $(0, 1]$ and introduce the new variable $\mu := 1 - \rho \geq 0$. It follows that $\mu \in [0, 1)$ and that

$$\langle D^{-1}(y_{\mathrm{d}} - Cx), y_{\mathrm{d}}\rangle \leq \frac{\varphi(\mu y_{\mathrm{d}}) - \varphi(y_{\mathrm{d}})}{1-\mu} \text{ for all } \mu \in [0, 1) \ .$$

Multiplying both sides by $1 - \mu > 0$ yields

$$\begin{aligned}&\langle D^{-1}(y_{\mathrm{d}} - Cx), (1-\mu)y_{\mathrm{d}}\rangle \leq \\ &\qquad \varphi(\mu y_{\mathrm{d}}) - \varphi(y_{\mathrm{d}}) \text{ for all } \mu \in [0, 1] \ .\end{aligned}$$

Furthermore, each element of S can be represented as $\sigma = \mu y_{\mathrm{d}} \in S$ for some $\mu \in [0, 1]$, therefore

$$\langle D^{-1}(y_{\mathrm{d}} - Cx), \sigma - y_{\mathrm{d}}\rangle + \varphi(\sigma) - \varphi(y_{\mathrm{d}}) \geq 0 \text{ for all } \sigma \in S \ .$$

That is, y_{d} is a solution of the hemivariational inequality (8) when $x \in \Omega_{\mathrm{d}}$ and, considering the uniqueness of solutions, the output y must be equal to y_{d}. Conversely, taking $y = y_{\mathrm{d}}$ from (8) we have that

$$-\langle D^{-1}(y_{\mathrm{d}} - Cx), \sigma - y_{\mathrm{d}}\rangle \leq \varphi(\sigma) - \varphi(y_{\mathrm{d}})$$

holds for all $\sigma \in S$. From (14) we have that $\sigma = \mu y_{\mathrm{d}}$ for some $\mu \in [0, 1]$, so the previous inequality is equivalent to

$$\langle D^{-1}(y_{\mathrm{d}} - Cx), y_{\mathrm{d}}\rangle \leq \frac{\varphi(\mu y_{\mathrm{d}}) - \varphi(y_{\mathrm{d}})}{1-\mu}$$

for all $\mu \in [0, 1)$. Hence, making the change of variables $\rho = 1 - \mu$ and considering the increasing property of the map $t \mapsto \frac{\varphi(v+td)-\varphi(v)}{t}$, the desired result is obtained.

It remains to show that (even in the presence of the external perturbation ν), the system state x enters the interior of the set Ω_{d} in finite time and remains therein for all future time. In terms of the equilibrium x_*, we have from (13) that $x_* \in \Omega_{\mathrm{d}}$. We will prove that, for some $\delta > 0$ small enough, there exists an ellipsoid $\mathcal{E} = \{x \in \mathbb{R}^n : (x - x_*)^\top P(x - x_*) \leq \delta\} \subset \operatorname{int}\Omega_{\mathrm{d}}$ around x_* that is attractive and invariant.

Considering the dissipation equation (12), it is clear that $\eta = u - \bar{u}$ is well defined, where $\bar{u}$ is the basic IDA control from Lemma 2, and $u \in \partial\Phi(y)$. Then, equation (12) transforms into

$$\begin{aligned}\dot{H}_2(x) = \frac{1}{2}&\begin{bmatrix}(x - x_*)^\top & -(u - \bar{u})^\top\end{bmatrix} \mathcal{R} \begin{bmatrix} x - x_* \\ -(u - \bar{u})\end{bmatrix} \\ &- (y - y_{\mathrm{d}})^\top(u - \bar{u}) + (x - x_*)^\top P B_{\mathrm{v}}\nu \ ,\end{aligned}$$

where the term $-(y - y_{\mathrm{d}})^\top(u - \bar{u})$ is negative for all $y \neq y_{\mathrm{d}}$ (i.e., for all $x \notin \Omega_{\mathrm{d}}$). Indeed, we have from (13) and Lemma 2 that

$$-\langle \bar{u}, y_{\mathrm{d}}\rangle < \mathcal{D}\varphi(y_{\mathrm{d}}, -y_{\mathrm{d}}) \ ,$$

and from the definition of a subdifferential we have

$$u \in \partial\Phi(y) \Leftrightarrow \varphi(\sigma) - \varphi(y) \geq \langle u, \sigma - y\rangle \text{ for all } \sigma \in S \ .$$

Specifically, for $\sigma = y_{\mathrm{d}}$ we obtain $-\langle u, y - y_{\mathrm{d}}\rangle \leq \varphi(y_{\mathrm{d}}) - \varphi(y)$. Moreover, for all $y \in S \setminus \{y_{\mathrm{d}}\}$ we can write $y = \mu y_{\mathrm{d}}$ with $\mu \in [0, 1)$. Thus,

$$\begin{aligned}-(y - y_{\mathrm{d}})^\top(u - \bar{u}) &\leq \varphi(y_{\mathrm{d}}) - \varphi(\mu y_{\mathrm{d}}) - (1-\mu)y_{\mathrm{d}}^\top\bar{u} \\ &< \varphi(y_{\mathrm{d}}) - \varphi(\mu y_{\mathrm{d}}) \\ &\quad + (1-\mu)\inf_{\rho>0}\frac{\varphi(y_{\mathrm{d}} - \rho y_{\mathrm{d}}) - \varphi(y_{\mathrm{d}})}{\rho} \\ &\leq \varphi(y_{\mathrm{d}}) - \varphi(\mu y_{\mathrm{d}}) \\ &\quad + (1-\mu)\frac{\varphi(y_{\mathrm{d}} - \rho y_{\mathrm{d}}) - \varphi(y_{\mathrm{d}})}{\rho}\end{aligned}$$

for all $\rho > 0$. Setting $\rho = 1 - \mu > 0$ we obtain $-(y - y_{\mathrm{d}})^\top(u - \bar{u}) < 0$ for all $y \neq y_{\mathrm{d}}$.

From (10b) we have that u must satisfy $u = D^{-1}(Cx - y)$. Substituting u and $\bar{u}$ in (12) and applying the Lambda inequality to the term $(x - x_*)^\top P B_{\mathrm{v}}\nu$ (see, e.g., Section 12.1 in [30]), we have

$$\begin{aligned}\dot{H}_2 \leq -\frac{1}{2}&w^\top R w - (y - y_{\mathrm{d}})^\top(u - \bar{u}) + \\ &(x - x_*)^\top\Lambda(x - x_*) + \nu^\top B_{\mathrm{v}} P\Lambda^{-1}P B_{\mathrm{v}}\nu \ ,\end{aligned}$$

where $\Lambda = \Lambda^\top > 0$ and

$$\begin{aligned}w^\top &= \begin{bmatrix}(x - x_*)^\top & (y - y_{\mathrm{d}})^\top D^{-\top}\end{bmatrix} \\ R &= -\begin{bmatrix} I & -C^\top D^{-\top} \\ 0 & I\end{bmatrix} \mathcal{R} \begin{bmatrix} I & 0 \\ -D^{-1}C & I\end{bmatrix} \ .\end{aligned}$$

It follows that $R > 0$. Now, setting Λ such that

$$R_\Lambda = R - \begin{bmatrix}\Lambda & 0 \\ 0 & 0\end{bmatrix} > 0 \ ,$$

we have

$$\dot{H}_2(x) \leq -\frac{1}{2}w^\top R_\Lambda w - (y - y_{\mathrm{d}})^\top(u - \bar{u}) + \nu^\top B_{\mathrm{v}} P\Lambda^{-1}P B_{\mathrm{v}}\nu$$

and therefore

$$\dot{H}_2(x) \leq -\frac{1}{2}\lambda_{\min}(R_\Lambda)\|w\|^2 + \lambda_{\max}(B_{\mathrm{v}}P\Lambda^{-1}PB_{\mathrm{v}})\|\nu\|^2$$
$$\leq -\frac{1}{2}\lambda_{\min}(R_\Lambda)\|x - x_*\|^2$$
$$+ \lambda_{\max}(B_{\mathrm{v}}P\Lambda^{-1}PB_{\mathrm{v}})\|\nu\|^2 .$$

Recall that we are establishing the stability of the ellipsoid $\mathcal{E}$, defined above. We have that, for all $x \notin \mathcal{E}$,

$$\|x - x_*\|^2 > \frac{\delta}{\lambda_{\max}(P)} ,$$

and therefore, if ν satisfies

$$\|\nu\|^2 \leq \frac{\delta\lambda_{\min}(R_\Lambda)}{2\lambda_{\max}(P)\lambda_{\max}(B_{\mathrm{v}}^\top P\Lambda^{-1}PB_{\mathrm{v}})} = \mathcal{B}^2 ,$$

we conclude that $\dot{H}_2 < 0$ for all $x \notin \mathcal{E}$. That is, the set $\mathcal{E}$ is attractive and invariant [31].

Finally, finite-time convergence of the output is obtained automatically from the proof. Namely, $\mathcal{E} \subset \operatorname{int}\Omega_{\mathrm{d}}$ together with attractivity and invariance of $\mathcal{E}$ implies that there exists a time $t^* < \infty$ such that the state will cross the boundary of Ω_{d} and will remain inside of Ω_{d} for all $t > t^*$. □

Figures 2 and 3 show an example of how the term Cx converges to the interior of the set $\{y_{\mathrm{d}}\}+N_S(y_{\mathrm{d}})$ in the output space when $S = \operatorname{conv}\{0, y_{\mathrm{d}}\}$ and $S = \operatorname{conv}\{[0,0],[0,y_{\mathrm{d}1}]\} \times \operatorname{conv}\{[0,0],[0,y_{\mathrm{d}_2}]\}$, respectively, and when $\varphi = 0$, $D = I_n$ and $m = 2$. Note that, $Cx - y_{\mathrm{d}} \in N_S(y_{\mathrm{d}})$ is equivalent to $y_{\mathrm{d}} = \operatorname{Proj}_S(Cx)$ and from (6) we obtain $y = y_{\mathrm{d}}$.

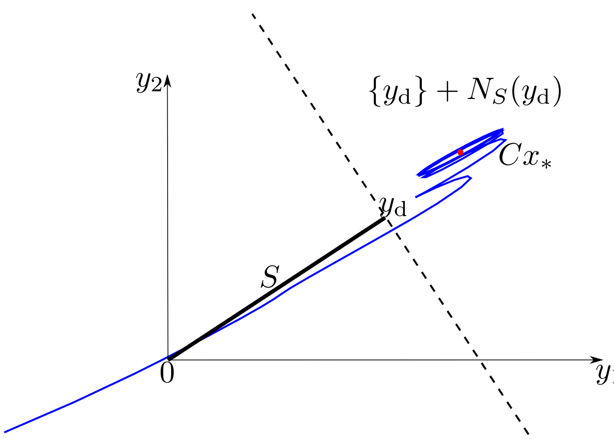

Fig. 2. Trajectory of Cx converging to interior of $\{y_d\} + N_S(y_{\mathrm{d}})$ with the multivalued control $u \in N_S(y)$ and $S = \operatorname{conv}\{0, y_{\mathrm{d}}\}$. This implies that y converges to y_{d}.

Remark 6. From Figure 2 it is possible to see that, if x_* satisfies the condition (13) for $y_{\mathrm{d}} \in S$, then we can achieve robust output regulation for any other desired value $\bar{y}_{\mathrm{d}}$ in the relative interior of S by redefining the set S to $\bar{S} = \operatorname{conv}\{0, \bar{y}_{\mathrm{d}}\}$. Moreover, in a more general setting, condition (13) allows us to attack the problem of robust tracking in the following way. Let $y_{\mathrm{d}}(t)$ be the desired reference signal. If, for all values of the function $y_{\mathrm{d}} : \mathbb{R} \to \mathbb{R}^m$, condition (13) is satisfied together with the bound in $\nu(t)$, then robust output tracking is possible as shown in Example 1 below.

Remark 7. A similar result can be obtained (with possibly different bounds in the external perturbation and different condition in x_*), if we change the form of the set S. For example, for $\Phi(y) = \psi_S(y)$ a possible set S could be as the one given in Figure 3, where the point y_{d} is still at the boundary of S and the interior of the normal cone to S at y_{d} is not empty. The details are left to the reader.

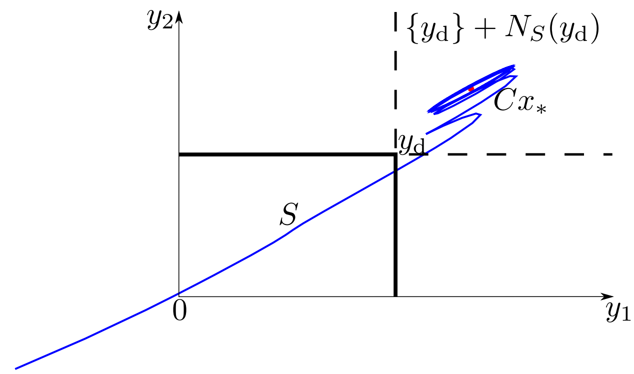

Fig. 3. Trajectory Cx converging to interior of $\{y_d\} + N_S(y_{\mathrm{d}})$ with the multivalued control $u \in N_S(y)$ and $S = \operatorname{conv}\{[0,0],[0,y_{\mathrm{d}1}]\} \times \operatorname{conv}\{[0,0],[0,y_{\mathrm{d}_2}]\}$.

Remark 8. When $\varphi \equiv 0$ and D is symmetric, the control law (7) yields robust output regulation without apparent knowledge of a disturbance model. That is, the necessary conditions stated by the internal model principle (IMP) [32] do not seem to hold. A possible explanation is that, although the control law is continuous, the vector field associated to the closed-loop system is *nonsmooth*. This transgresses a fundamental hypothesis in the IMP literature. The IMP in the nonsmooth setting is an under-explored topic that deserves further research.

It is worth noting that, although the controller alone is set-valued (cf. (4)), the interconnected system (3) admits a unique control input $u \in \partial\Phi(y)$. This is easily seen when $\varphi = 0$ and D is symmetric and positive definite. In such case, the controller is given by the inclusion $u \in N_C(y)$, but the interconnection only admits the single-valued expression (7).

A similar result is obtained in the general case where D is not necessarily symmetric and φ is an arbitrary proper, convex, lower semicontinuous function. More precisely, it can be shown that the solution of the hemivariational inequality (8) (i.e., y^*), determines a unique control $u = D^{-1}(Cx - y^*)$ — which under our uncertainty assumptions is not directly implementable as it stands (see Remarks 3 and 4).

Before finishing this section, we state the following proposition showing the continuity of the resulting controller.

Proposition 1. *The control input η resulting from the closed-loop system* (3) *and* (4) *is Lipschitz continuous with respect to the system state x.*

Proof. Since $\eta = u - \bar{u}$ with $\bar{u}$ as in Lemma 2 and $u \in \partial\Phi(y)$, it is clear that η and u will have the same continuity properties. Since $u \in \partial\Phi(y)$ must satisfy (3b), the hemivariational inequality (8) must be satisfied as well. Therefore, for any

$y_1 \in S$ and $y_2 \in S$,

$$\langle D^{-1}(Cx_1 - y_1), \sigma_1 - y_1\rangle \le \varphi(\sigma_1) - \varphi(y_1)$$
$$\langle D^{-1}(Cx_2 - y_2), \sigma_2 - y_2\rangle \le \varphi(\sigma_2) - \varphi(y_2)$$

are satisfied for all $\sigma_1 \in S$ and $\sigma_2 \in S$. Taking $\sigma_1 = y_2$ and $\sigma_2 = y_1$ and adding both inequalities yields to

$$\langle D^{-1}(y_1 - y_2), y_1 - y_2\rangle \le \langle D^{-1}C(x_1 - x_2), y_1 - y_2\rangle$$

for all $y_i \in S$ and $x_i \in \mathbb{R}^n$, $i \in \{1, 2\}$. From the positiveness assumption about D, we have

$$\|y_1 - y_2\| \le L\|x_1 - x_2\|$$

with $L = \|D^{-1}C\|/\lambda_{\min}(D_S^{-1})$. Finally, the control u satisfies

$$\begin{aligned}\|u_1 - u_2\| &= \|D^{-1}(Cx_1 - y_1) - D^{-1}(Cx_2 - y_2)\| \\ &\le \|D^{-1}C\|\|x_1 - x_2\| + \|D^{-1}\|\|y_1 - y_2\| \\ &\le \tilde{L}\|x_1 - x_2\|\end{aligned}$$

with $\tilde{L} = \left(1 + \frac{\|D^{-1}\|}{\lambda_{\min}(D_S^{-1})}\right)\|D^{-1}C\|$. $\square$

IV. Implementation issues and examples

A. Regularization

Up to this point we have shown that, whenever $u \in \partial\Phi(y)$, the membership of y to the set S and the robust output regulation are assured. Our next step is to develop a way to recover an explicit expression for the values of the control input u in terms of the measured output y alone.

Note that exact values of u can be computed by solving the hemivariational inequality (8) at each time instant t and making use of (10b), but this approach requires knowledge of the system parameters and state x. It is worth noting that searching for continuous selections does not yield the desired features either. For example, in the case of $\varphi = 0$, a continuous selection of the multifunction $\partial\Phi(\cdot) = N_S(\cdot)$ is $u = 0$, (in fact $u = 0$ is the unique continuous selection). However, with that control the storage function of (10) is given by H_1 with minimum at $\bar{x}$ and, consequently, neither robust output regulation nor $y \in S$ are in general obtained. Similar results can be obtained when $\varphi \in \mathcal{C}^1$, since $u = \nabla\varphi(\cdot)$ is always a continuous selection of $\partial\Phi(\cdot)$.

Instead of looking for continuous selections of $\partial\Phi(\cdot)$, we will focus on a regularization of $\operatorname{Graph}\partial\Phi$ in the sense used in [25]. More precisely,

$$\tilde{u} - \nabla\varphi(y) \in N_S(y - \varepsilon[\tilde{u} - \nabla\varphi(y)]) \tag{15}$$

is a regularization of the inclusion $u \in \partial\Phi(y)$. Note that, for $\varepsilon = 0$, we recover $\tilde{u} \in \partial\Phi(y)$ (this is because $\partial\Phi(y) = \nabla\varphi(y) + N_S(y)$). Moreover, with the previous definition we allow initial outputs y not necessarily in S. Instead we now require $y \in \{\varepsilon[\tilde{u} - \nabla\varphi(y)]\} + S$.

The well-posedness of inclusion (15) together with a single-valued expression for $\tilde{u}$ are established below in Theorem 2. The following lemma will be useful when proving it.

Lemma 3. *The map $f : \mathbb{R}^m \to \mathbb{R}^m$ given by*

$$f(z) := (I + \varepsilon D^{-1})^{-1} z\ ,$$

where $D^{-1} > 0$ is a contraction for all $\varepsilon > 0$.

Proof. After defining

$$\zeta = f(z)\ ,$$

we have $(I + \varepsilon D^{-1})\zeta = z$. Direct computation gives

$$\begin{aligned}\|z_1 - z_2\|^2 &= \|(I + \varepsilon D^{-1})(\zeta_1 - \zeta_2)\|^2 \\ &= \|\zeta_1 - \zeta_2\|^2 + \varepsilon(\zeta_1 - \zeta_2)^\top D_S^{-1}(\zeta_1 - \zeta_2) \\ &\quad + \varepsilon^2(\zeta_1 - \zeta_2)^\top D^{-\top}D^{-1}(\zeta_1 - \zeta_2) \\ &\ge \|\zeta_1 - \zeta_2\|^2 + \varepsilon\lambda_{\min}(D_S^{-1})\|\zeta_1 - \zeta_2\|^2 \\ &\quad + \varepsilon^2\lambda_{\min}(D^{-\top}D^{-1})\|\zeta_1 - \zeta_2\|^2\ .\end{aligned}$$

Therefore,

$$\|f(z_1) - f(z_2)\| \le \frac{1}{\beta}\|z_1 - z_2\|$$

with

$$\beta = \sqrt{1 + \varepsilon\lambda_{\min}\left(D^{-1} + D^{-\top}\right) + \varepsilon^2\lambda_{\min}\left(D^{-\top}D^{-1}\right)}\ .$$

$\square$

Theorem 2. *Let φ be a strictly convex, lower semicontinuous $\mathcal{C}^1$ function that satisfies*

- *$\varphi(y) \ge \varphi(0)$ for all $y \in S$.*
- *$\nabla\varphi : \mathbb{R}^m \to \mathbb{R}^m$ is Lipschitz continuous with constant L such that*
$$L < \lambda_{\min}\left(\frac{D_S^{-1}}{2}\right)\ .$$

Then, for $\varepsilon > 0$ sufficiently small, the regularized control $\tilde{u}$ can be expressed as

$$\tilde{u} = \frac{y - \operatorname{Proj}_S(y)}{\varepsilon} + \nabla\varphi(y). \tag{16}$$

Furthermore, the map $y \mapsto \tilde{u}$ is passive.

Proof. From (15) we have that, for all $\sigma \in S$,

$$0 \ge \langle \tilde{u} - \nabla\varphi(y), \sigma - y + \varepsilon[\tilde{u} - \nabla\varphi(y)]\rangle\ . \tag{17}$$

Multiplying by $\varepsilon > 0$ and adding and subtracting y on the left-hand side of the inner product we obtain

$$0 \ge \langle y - y + \varepsilon[\tilde{u} - \nabla\varphi(y)], \sigma - y + \varepsilon[\tilde{u} - \nabla\varphi(y)]\rangle\ .$$

Therefore,

$$y - \varepsilon[\tilde{u} - \nabla\varphi(y)] = \operatorname{Proj}_S(y)\ ,$$

from which we obtain (16). Now we show that the interconnection of the plant (10a)–(10b) with the regularized control (16) is well-posed. It is easy to see that well-posedness of the closed-loop system is equivalent to proving that, for any state $x \in \mathbb{R}^n$, the equations

$$\begin{aligned}\tilde{u} &= \frac{y - \operatorname{Proj}_S(y)}{\varepsilon} + \nabla\varphi(y) \\ \tilde{u} &= D^{-1}(Cx - y)\end{aligned}$$

have a unique solution. Proceeding with the substitution of the second equation and after some manipulations we have

$$\begin{aligned}y = \left(I + \varepsilon D^{-1}\right)^{-1}&\left[\operatorname{Proj}_S(y) - \varepsilon\nabla\varphi(y) + \varepsilon D^{-1}Cx\right] \\ &= (f \circ g)(y)\ ,\end{aligned}$$

with f as in Lemma 3 and $g : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$g(z) = \mathrm{Proj}_S(z) - \varepsilon \nabla\varphi(z) + \varepsilon D^{-1}Cx \ .$$

We argue that the composition mapping $f \circ g$ is a contraction for ε sufficiently small. Indeed, making use of Lemma 3 we have that

$$\begin{aligned}\| \left(f \circ g\right)(y_1) - \left(f \circ g\right)(y_2) \| &\le \frac{1}{\beta(\varepsilon)} \|g(y_1) - g(y_2)\| \\ &\le \frac{1+\varepsilon L}{\beta(\varepsilon)} \|y_1 - y_2\|\end{aligned}$$

where $\beta(\varepsilon)$ is defined in the proof of Lemma 3. Note that the term $(1+\varepsilon L)/\beta(\varepsilon)$ is equal to 1 for $\varepsilon = 0$ and

$$\frac{d}{d\varepsilon}\left(\frac{1+\varepsilon L}{\beta(\varepsilon)}\right)\bigg|_{\varepsilon=0} = L - \lambda_{\min}\left(\frac{D_S^{-1}}{2}\right) < 0 \ ,$$

i.e., the term $(1+\varepsilon L)/\beta(\varepsilon)$ is strictly decreasing in a neighbourhood of $\varepsilon = 0$ and thus it is less than 1 for ε sufficiently small. Therefore, $f \circ g$ is a contraction and the interconnection is well-posed.

It only rests to prove the passivity property of $(\tilde{u}, y)$. From (17) we have, for $\sigma = 0 \in S$,

$$\langle \tilde{u}, y \rangle \ge \langle \nabla\varphi(y), y \rangle + \varepsilon \|\tilde{u} - \nabla\varphi(y)\|^2 \ .$$

Note that for $\varepsilon = 0$ we have $y \in S$ and from the strict convexity assumption [24, p. 183],

$$\langle \nabla\varphi(y), y \rangle > \varphi(y) - \varphi(0) \ge 0 \text{ for all } y \in S \ .$$

In other words, we have

$$\lim_{\varepsilon \downarrow 0} \langle \nabla\varphi(y), y \rangle + \varepsilon \|\tilde{u} - \nabla\varphi(y)\|^2 > 0 \ .$$

Consequently, $\langle \tilde{u}, y \rangle \ge 0$ for some $\varepsilon > 0$ sufficiently small. $\square$

Remark 9. Note that Theorem 2 is still true if we change the first assumption by $\varphi(y) \ge \varphi(0)$ for all $y \in \mathrm{Dom}\,\varphi$ with φ a convex function.

From Theorem 2 we have that the regularized control (15) is in fact single-valued, Lipschitz continuous and independent of the system parameters and state (cf. (16)).

B. Example 1

Consider the circuit described by the diagram of Figure 4. We wish to regulate the outputs y_1 and y_2 to a desired value y_d. Taking as state variables the fluxes in inductors and charges in capacitors, we have the following state-space representation:

$$\dot{x} = \begin{bmatrix} -\frac{1}{R_1C_1} & -\frac{1}{L_1} & -\frac{1}{R_1C_2} & 0 \\ \frac{1}{C_1} & -\frac{R_2+R_3+R_{L_1}}{L_1} & 0 & \frac{R_3}{L_2} \\ -\frac{1}{R_1C_1} & 0 & -\frac{1}{R_1C_2} & -\frac{1}{L_2} \\ 0 & \frac{R_3}{L_1} & \frac{1}{C_2} & -\frac{R_3+R_{L_2}}{L_2} \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ R_{L_1} & 0 \\ 0 & 0 \\ 0 & R_{L_2} \end{bmatrix} u + \begin{bmatrix} \frac{1}{R_1} \\ 0 \\ \frac{1}{R_1} \\ 0 \end{bmatrix} v \tag{18a}$$

$$y = \begin{bmatrix} 0 & \frac{R_{L_1}}{L_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{R_{L_2}}{L_2} \end{bmatrix} x + \begin{bmatrix} R_{L_1} & 0 \\ 0 & R_{L_2} \end{bmatrix} u \tag{18b}$$

where $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\top$ are the charge in capacitor C_1, flux in inductor L_1, charge in capacitor C_2 and flux in inductor L_2, respectively, $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^\top$ are the control inputs (currents) and $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^\top$ are the voltages in resistances R_{L_1} and R_{L_2}, respectively. Assume that we want to steer the outputs to $y_d = \begin{bmatrix} 1 & f(t) \end{bmatrix}^\top$, where $f(t)$ is a sawtooth wave function with amplitude 0.5 and frequency of 2 Hz.

The system is passive because it is the result of the interconnection of passive elements. Values of system parameters are $R_1 = R_2 = R_3 = 1\Omega$, $R_{L_1} = 2\Omega$, $R_{L_2} = 3\Omega$, $L_1 = 1$H, $L_2 = 2$H, $C_1 = 1$F, $C_2 = 3$F, $v = 10 + 50\sin(t)\,\mathrm{sign}(\sin(\pi t))$. Taking the convex function $\varphi = 0$ (i.e., $u \in N_S(y)$), simple algebra shows that condition (13) is equivalent to

$$\langle D^{-1}(y_\mathrm{d} - Cx_*), y_\mathrm{d} \rangle = -4 - 4f(t) + \frac{5}{6}f(t)^2 \ ,$$

which is negative for values of $f(t) \in (-0.849, 5.649)$. The implemented control takes the form (16) with $\varepsilon = 1 \times 10^{-3}$ and S the convex, time-varying set

$$S(t) = \mathrm{conv}\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ f(t) \end{bmatrix} \right\} \ .$$

Figure 5 shows the convergence of the output to the desired reference, even in the presence of the external perturbation v. Moreover, it is easy to see that the condition $y \in S$ is satisfied. The computed control input is shown in Figure 6.

C. Example 2

Consider the following affine system

$$\dot{x} = Ax + B_\mathrm{u}u_1 + B_\mathrm{v}v \tag{19a}$$
$$y_1 = Cx + Du_1 \tag{19b}$$

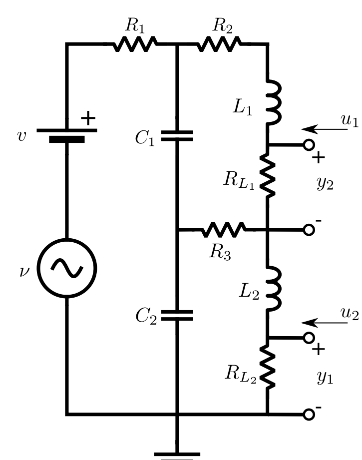

Fig. 4. Circuit diagram of Example 1, where the goal is to regulate the voltage at the outputs y_1 and y_2.

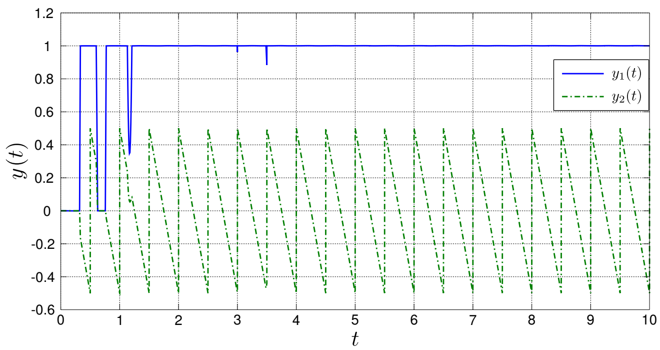

Fig. 5. Output response of plant (18) with the regularized control (16). The picture shows convergence to the desired output $y_{\rm d}$ (with $\varepsilon = 1 \times 10^{-3}$ and $\varphi = 0$) subject to the perturbation $v(t) = 10 + 50 \sin(t)\,\mathrm{sign}(\sin(\pi t))$.

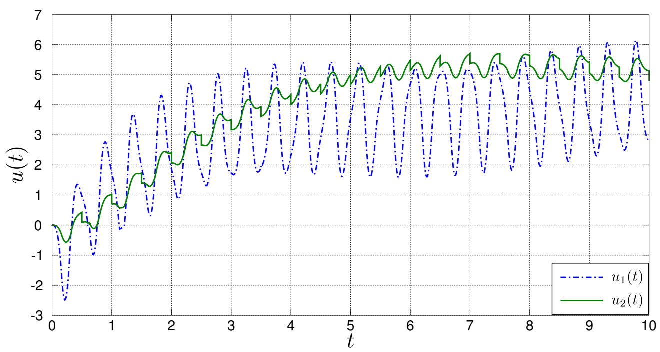

Fig. 6. Time history of the regularized control (16) with $\varepsilon = 1 \times 10^{-3}$ and $\varphi = 0$.

with

$$A = \begin{bmatrix} -3.7036 & 1.9043 & -0.9735 & 0.4164 \\ -0.2421 & -5.1187 & -0.0478 & 0.0269 \\ -0.9915 & -1.0461 & -5.5232 & -0.5318 \\ 0.4376 & 2.1467 & -0.5948 & -3.6545 \end{bmatrix},$$

$$B_{\rm u} = \begin{bmatrix} -0.6918 & -1.4410 \\ 0.8580 & 0.5711 \\ 1.2540 & -0.3999 \\ -1.5937 & 0.6900 \end{bmatrix}, B_{\rm v} = \begin{bmatrix} 0.8147 & 1.4345 \\ 0.9058 & 2.5464 \\ -0.1270 & 0 \\ 0.9134 & -1.0453 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.0652 & 0.4889 & 0.6820 & 0.9166 \\ 0.7134 & 0.6677 & 0.1996 & 0.8659 \end{bmatrix},$$

$$D = \begin{bmatrix} 1.0823 & 0.3899 \\ -0.1315 & 0.088 \end{bmatrix},$$

where the external perturbation signal $v(t)$ is decomposed as

$$v(t) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{20}$$

with $f_1(t)$ a sinusoidal function with amplitude 2 and frequency of 10 Hz and $f_2(t)$ corresponds to a sawtooth wave with amplitude 3 and frequency of π Hz. Suppose that we want to regulate the output to the set-point $y_{\rm d} = \begin{bmatrix} -1 & 2 \end{bmatrix}$.

Let us verify the assumptions of Theorem 1. The equilibrium point x_* is

$$x_* = \begin{bmatrix} 2.7809 & 0.1184 & -0.2779 & 0.4877 \end{bmatrix}^\top .$$

and it satisfies

$$\langle D^{-1}(y_{\rm d} - Cx_*), y_{\rm d} \rangle = -9.2810 .$$

Taking, for example, the convex function

$$\varphi(y) = \log\left(e^{y_1} + e^{y_2}\right) \ ,$$

which is proper and $\mathcal{C}^1$, we have that

$$\mathcal{D}\varphi(y_{\rm d}, -y_{\rm d}) = -\langle \nabla\varphi(y_{\rm d}), y_{\rm d} \rangle = -1.8577 .$$

Condition (13) is satisfied. Using the SDPT3 software to solve (2) we obtain

$$P = \begin{bmatrix} 1.8765 & 1.8706 & -0.5249 & 1.3338 \\ 1.8706 & 3.8984 & -0.4599 & 0.9207 \\ -0.5249 & -0.4599 & 2.4211 & 0.4920 \\ 1.3338 & 0.9207 & 0.4920 & 2.0056 \end{bmatrix},$$

which is positive definite with eigenvalues 0.23, 1.54, 2.74, 5.69. Figure 7 shows the output response for a regularized control $\tilde{u}$ with $\varepsilon = 1 \times 10^{-4}$, where finite-time convergence towards the desired set-point can be verified despite the external and parametric disturbances of the system. Control and state trajectories are shown in Figures 8 and 9, respectively.

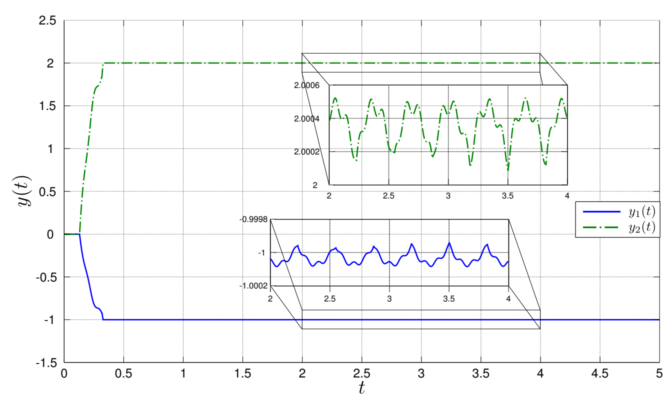

Fig. 7. Time history of the output for plant (19). It illustrates the convergence to the desired value $y_{\rm d} = [-1 \quad 2]^\top$.

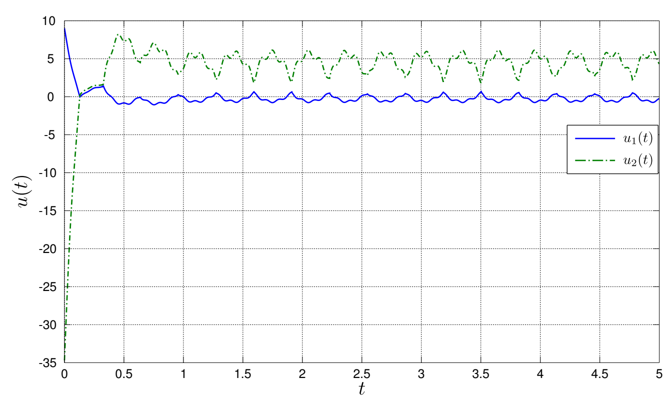

Fig. 8. Time history of the regularized control (16) applied to the plant (19) with $\varepsilon = 1 \times 10^{-4}$ and $\varphi = \log(e^{y_1} + e^{y_2})$.

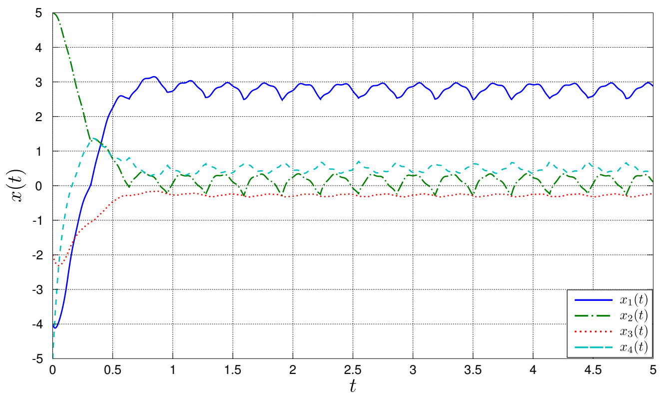

Fig. 9. Time histories of the state of regulated plant (19) with the regularized control (16) (with $\varepsilon = 1 \times 10^{-4}$ and $\varphi = \log(e^{y_1} + e^{y_2})$) and perturbation (20).

V. Conclusions

This paper presents an extension (for the m-dimensional case) of the multivalued control presented in [25]. Moreover, more general multivalued functions of the form $u \in \partial\Phi(y)$ are considered, assuring finite-time convergence together with robust output regulation in the face of parametric and external (bounded) disturbances.

The effect of the multivalued control relies directly on the dissipation term modifying the rate of convergence of the storage function H_2 and implying the convergence of x to x_*, leaving interconnection matrix J unchanged.

Among the main assumptions considered, the fact that D is invertible plays an essential role. The case without D (i.e. $y = Cx$) is a topic of further research.

The implemented control (16) acts in fact as a high gain controller when $y \notin S$ and coincides with the continuous selection of $\partial\Phi(y)$ when $y \in S$. However, since the output contains a feedthrough component of the input, the high gain does not result in arbitrary large controls. That is, the control converges to a bounded, well-defined value as $\varepsilon \to 0$. It is worth noting that the resulting controller is passive and independent of the system parameters and of the system state.

The well-suited structure of port-Hamiltonian systems together with passivity opens the opportunity to investigate the robust output regulation problem in the nonlinear setting.

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Fernando Castaños (S'06-M'10) received the B.Eng. in Electric and Electronic Engineering from Universidad Nacional Autónoma de México (UNAM). He received the M.Eng. in Control Engineering from UNAM and the Ph.D degree from Université Paris-Sud XI. He was a postdoctoral fellow at McGill's Center for intelligent machines for two years. He joined the Automatic Control Department, Cinvestav, Mexico, in 2011. His research interests are passivity-based control, nonlinear control, port-Hamiltonian systems, robust control and variable structure systems.

Félix A. Miranda-Villatoro (S'15) is a Ph.D student at the Automatic Control Department, CINVESTAV-IPN, Mexico. He received the B.Eng. degree in Electronic Engineering from the Universidad Autónoma del Estado de México (UAEMex), in 2010. After that, he obtained the M.Sci. degree from CINVESTAV-IPN in 2012. His research interests include robust control, differential inclusions, and the application of nonsmooth tools in control theory.