Integral Sliding Mode Control for Nonlinear Systems with Matched and Unmatched Perturbations

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***Abstract*—We consider the problem of designing an integral sliding mode controller to reduce the disturbance terms that act on nonlinear systems with state-dependent drift and input matrix. The general case of both, matched and unmatched disturbances affecting the system is addressed. It is proved that the definition of a suitable sliding manifold and the generation of sliding modes upon it guarantees the minimization of the effect of the disturbance terms, which takes place when the matched disturbances are completely rejected and the unmatched ones are not amplified. A simulation of the proposed technique, applied to a dynamically feedback linearized unicycle, illustrates it effectiveness, even in presence of nonholonomic constraints.**

***Index Terms*—Sliding mode control, uncertain systems, integral sliding mode, disturbance reduction.**

I. Introduction

Sliding mode control [1], [2] is a robust technique for the control of nonlinear systems. The most positive feature of sliding mode control consists in the complete compensation of the so-called *matched disturbances* (i.e. disturbances acting on the control input channel) when the system is in the *sliding phase* and a *sliding mode* is enforced. This latter takes place when the state is on a suitable subspace of the state space, called *sliding manifold*. The compensated dynamics become insensitive to matched disturbances and uncertainties under sliding mode control. The price for this insensitivity is control chattering and a reaching phase, during which the system dynamics are vulnerable to disturbances/uncertainties.

The integral sliding mode (ISM) technique was first proposed in [3], [4] as a solution to the reaching phase problem for systems with matched disturbances only. The ISM control can also be regarded as a way to combine the use of the sliding mode controller with that of another controller (called *high level controller* in the following). The latter aims at stabilizing the nominal system. Systems compensated with this type of controllers are of full order, i.e., of order equal to the order of uncompensated system. When the system is subjected to external bounded perturbations, it is natural to try to compensate such perturbations by means of an auxiliary control that retains the effect of the controller designed for unperturbed system. The sliding mode-based auxiliary controller that compensates the perturbation from the very beginning of the control action, while retaining the order of uncompensated system, is the ISM controller. This technique has been deeply studied in the last years (see e.g., [5]–[12] and the references therein).

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Recently, the problem of analyzing how to minimize the disturbance terms using ISM has been taken into account for systems with a nonlinear drift term and a constant input matrix [13]. This result has been used also in connection with other control strategies, for example with model predictive control in [14], allowing the use of less conservative high level controllers. Nevertheless, a very important class of systems have a state dependent input matrix, e.g., in mechanical systems the control is premultiplied by the inverse of the inertia matrix.

In this technical note, we consider the general class of nonlinear control-affine systems with both, matched and unmatched perturbations, and a state-dependent input matrix. Due to the appearance of partial derivatives in the state-dependent input matrix, the methodology of [13] cannot be directly applied. The contribution of this work consists in the definition of an integral sliding manifold which leads to the minimization of the effect of the disturbance terms also in this case (provided that some integrability conditions are met). Moreover, it is proved that in the particular case of systems in the so-called *regular form*, it is possible to use a constant matrix to define the sliding manifold, thus simplifying the design phase. Note that a preliminary version of the theoretical development of this paper, where only the particular case of systems in regular form were considered, can be found in [15].

The technical note is organized as follows: Section II introduces the considered class of systems and the control problem, while the proposed solution is analyzed in Section III. Simulation examples are reported in Section IV, and Section V draws the conclusions.

II. Problem statement

The system taken into consideration is of the form

$$\dot{x}(t) = f(x,t) + B(x)u(x,t) + \phi(x,t) \ , \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system with initial condition $x(t_0) = x_0$, $u \in \mathbb{R}^m$ is the control variable, $\phi(x,t) \in \mathbb{R}^n$ is an unknown vector representing modeling uncertainties and external disturbances, $f \in \mathbb{R}^n$ is a known nonlinear function, and $B \in \mathbb{R}^{n\times m}$ is a known full rank state-dependent matrix.

Assumption 1: The uncertain vector $\phi(x,t)$ is such that

$$\phi(x,t) \in \Phi,\ \Phi \triangleq \{v \in \mathbb{R}^n \ s.t.\ ||v||_2 \leq \Phi^{sup}\} \ ,$$

where Φ^{sup} is a constant scalar value.

The approach used in ISM control consists in splitting the control variable into two parts

$$u(x,t) = u_0(x,t) + u_1(x,t) \tag{2}$$

where the term $u_0(x,t)$ is generated by the high level controller (which can be designed according to any suitable design method), while $u_1(x,t)$ is a discontinuous control action designed to reject the disturbance terms, forcing the system state on a suitably designed sliding manifold $s(x,t) = 0$. In the following, the dependence of x and $\dot{x}$ on t is omitted in some cases, when it is obvious, for the sake of simplicity. The proposed integral sliding manifold can be defined as

$$s(x,t) = g(x) - z(x,t) \tag{3}$$

where $g(x) : \mathbb{R}^n \to \mathbb{R}^m$ is a nonlinear function, the total derivative of which is

$$\dot{g}(x) = G(x)\dot{x} \tag{4}$$

with

$$G(x) = \frac{\partial g(x)}{\partial x} \in \mathbb{R}^{m\times n} \tag{5}$$

representing the Jacobian matrix, and the integral term $z(x,t)$ is

$$z(x,t) = g(x_0) + \int_{t_0}^{t} G(x)\big(f(x,\tau) + B(x)u_0(x,\tau)\big)d\tau\,. \tag{6}$$

First of all, note that the system is in sliding mode at the initial time instant, i.e. $s(x_0,t_0) = 0$. Also, note that the integral sliding manifold in (3) is analogous to those proposed in [6] and [13], with the main difference that in our case the matrix $G(x)$ is not imposed to be constant, and is specifically designed to minimize the effect of the unmatched disturbances. The following assumption is required.

Assumption 2: $G(x)$ is such that

$$\text{rank}\,(G(x)B(x)) = m,\ \forall\ x \in \mathbb{R}^n. \tag{7}$$

The control law is designed relying on the *unit vector* approach [2], where one has

$$u_1(x,t) = -\rho(x,t)\frac{(G(x)B(x))^\top s(x,t)}{||(G(x)B(x))^\top s(x,t)||_2} \tag{8}$$

with $\rho \in \mathbb{R}$ a gain that guarantees the enforcing of the state motion on the sliding manifold, provided that Assumptions 1 and 2 hold.

In practice, the ideal aim of the ISM strategy would be to keep at zero the difference between the nominal and perturbed evolutions of the system. Taking into account that this could be done only when $n = m$, in the general case $m < n$ one has that only the projection of the difference between the two state evolutions along the projection onto the span of the rows of $G(x)$ can be kept equal to zero.

The uncertain vector ϕ for a system in form (1) can always be expressed by separating the matched disturbance ϕ_M from the unmatched one ϕ_U, as follows

$$\phi(x,t) = \phi_M(x,t) + \phi_U(x,t) \tag{9}$$

$$\phi_M(x,t) \triangleq B(x)B^+(x)\phi(x,t) \tag{10}$$

$$\phi_U(x,t) \triangleq B^\perp(x)B^{\perp+}(x)\phi(x,t) \tag{11}$$

where $B^\perp(x) \in \mathbb{R}^{n\times(n-m)}$ is a matrix with independent columns that span the null space of $B(x)$, i.e. $B^{\perp\top}(x)B(x) = \mathbf{0}_{(n-m)\times m}$, $\text{rank}(B^\perp(x)) = n-m$. Moreover, $B^+(x)$ is the left pseudo-inverse of $B(x)$, i.e., $B^+(x) \triangleq (B^\top(x)B(x))^{-1}B^\top(x)$. Analogously, we have $B^{\perp+}(x) \triangleq (B^{\perp\top}(x)B^\perp(x))^{-1}B^{\perp\top}(x)$. This separation principle relies on Proposition 1 in [13], which ensures that $I_n = B(x)B^+(x) + B^\perp(x)B^{\perp+}(x)$ for any full rank $B(x)$, being $I_n \in \mathbb{R}^{n\times n}$ an identity matrix. Thus, given the rank condition on $B(x)$, the decomposition (10)-(11) is without loss of generality. To determine the state equations when the state is confined to the sliding manifold, the equivalent control method [16] is used. This consists in forcing the derivative of $s(x,t)$ equal to zero, then determining the value of the equivalent control, and finally substituting it into the state equations. In the present case, the derivative of $s(x,t)$ is

$$\begin{aligned}\dot{s}(x,t) =& G(x)\dot{x} - \dot{z}(x,t)\\ =& G(x)\Big[f(x,t) + B(x)(u_0(x,t) + u_1(x,t) + B^+(x)\phi(x,t))\\ &+ \phi_U(x,t) - f(x,t) - B(x)u_0(x,t)\Big]\\ =& G(x)B(x)(u_1(x,t) + B^+(x)\phi(x,t)) + G(x)\phi_U(x,t)\end{aligned} \tag{12}$$

Then, the equivalent control, defined as the continuous control variable such that $\dot{s}(x,t) = 0$, results being

$$u_{1eq}(x,t) = -B^+(x)\phi(x,t) - (G(x)B(x))^{-1}G(x)\phi_U(x,t)\,. \tag{13}$$

Substituting this value into the system equation (1), one has that the matched disturbance is eliminated, and the trajectories of the system at the sliding manifold are given by

$$\dot{x}_{eq} = f(x,t) + B(x)u_0(x,t) + (I - B(x)(G(x)B(x))^{-1}G(x))\phi_U(x,t)$$

In conclusion, the action of the ISM control strategy has transformed the original uncertain term $\phi(x,t)$ into a new term

$$\phi_{eq}(x,t) \triangleq (I - B(x)(G(x)B(x))^{-1}G(x))\phi_U(x,t). \tag{14}$$

An optimal choice of the state-dependent matrix $G(x)$ would minimize this term. The goal of the reminder of this work is then to solve the following problem.

Problem 1: For system (1) fulfilling Assumptions 1 and 2, find a function $g^*(x)$ such that

$$\frac{\partial g^*(x)}{\partial x} \triangleq G^*(x) = \underset{G(x)\in\mathbb{R}^{m\times n}}{\arg\min}\ ||\phi_{eq}(x,t)||_2\,. \tag{15}$$

III. The proposed sliding manifold

A. Case 1: system in general form

A general result is hereafter introduced for the minimization of the equivalent disturbance (14) for system (1), when the ISM control strategy is applied. First of all, consider the distribution given by

$$\Delta(x) = \text{span}\left\{B_i^\perp(x)\right\},\ \ i = 1,\ldots,n-m \tag{16}$$

where $B_i^\perp$ stands for the $i-th$ column of $B^\perp$. We introduce the following assumption.

Assumption 3: $\Delta(x)$ is involutive, i.e.

$$\left[B_i^\perp(x), B_j^\perp(x)\right] = \frac{\partial B_j^\perp(x)}{\partial x}B_i^\perp(x) - \frac{\partial B_i^\perp(x)}{\partial x}B_j^\perp(x) \in \Delta(x),\quad \forall i,j = 1,\ldots,n-m \tag{17}$$

where $[\cdot,\cdot]$ is the Lie bracket of two vector fields.

Lemma 1: If Assumption 3 is fulfilled, there exists a function $\tilde{g}(x)$ such that

$$\frac{\partial\tilde{g}(x)}{\partial x} \triangleq \tilde{G}(x) = M(x)B^\top(x) \tag{18}$$

where $M(x) \in \mathbb{R}^{m\times m}$ is a full rank matrix. Note that (18) guarantees that Assumption 2 holds.

Proof: According to Frobenius' Theorem (see, e.g., [17]), the involutivity of $\Delta(x)$ is equivalent to the existence of m independent functions $\tilde{g}_i(x)$ such that

$$\frac{\partial\tilde{g}_i(x)}{\partial x}B_j^\perp(x) = 0 \qquad \text{for all}\, 1 \le i \le m\,,\, 1 \le j \le n-m$$

or, more compactly, $\tilde{G}(x)B^\perp(x) = 0$. Since the m columns of $\tilde{G}^\top(x)$ are independent, they span the orthogonal complement of $\Delta(x)$. That is,

$$\text{span}\left\{\tilde{G}_i^\top(x)\right\} = \left(\text{span}\left\{B_i^\perp(x)\right\}\right)^\perp. \tag{19}$$

Recall that the double orthogonal complement of a closed subspace is equal to the subspace itself [18, p. 118], so (19) is equivalent to $\text{span}\left\{\tilde{G}_i^\top(x)\right\} = \text{span}\{B_i(x)\}$. The columns of $\tilde{G}^\top(x)$ and $B(x)$ are bases of the same subspace and the matrix $M^\top(x)$ in (18) is simply the transformation matrix relating them. ■

Remark 1: The sufficiency part of the proof of Frobenius Theorem is constructive, thus providing an explicit procedure for finding $\tilde{g}(x)$ [17, pp. 24–26].

The main result of the paper is now formulated.

Theorem 1: Consider system (1) fulfilling Assumptions 1 and 3. Then, $\tilde{g}(x)$ solves Problem 1. Moreover, the resulting equivalent disturbance (14) is such that

$$||\phi_U(x,t)||_2 = \min_{G^*(x)\in\mathbb{R}^{m\times n}} ||\phi_{eq}(x,t)||_2 \tag{20}$$

Proof: First of all, note that Assumption 3 leads to the fulfillment of Lemma 1. Therefore, if $G(x) = \tilde{G}(x) = M(x)B^\top(x)$, it automatically follows that Assumption 2 is also fulfilled, because $B(x)$ is full rank. Then, if one sets $g(x) = \tilde{g}(x)$, it makes sense to face Problem 1 with the given assumptions. Analogously to [13], define

$$\varphi(x,t) \triangleq (G(x)B(x))^{-1}G(x)\phi_U(x,t)$$

Remarking that

$$||\phi_U(x,t) - B(x)\varphi(x,t)||_2 = ||I - B(x)(G(x)B(x))^{-1}G(x))\phi_U(x,t)||_2$$

we reformulate Problem 1 as the problem of finding $\varphi^*(x,t)$ such that

$$\varphi^*(x,t) = \underset{\varphi(x)\in\mathbb{R}^m}{\arg\min} ||\phi_U(x,t) - B(x)\varphi(x,t)||_2 \, .$$

According to the projection theorem in [18, p. 51], an explicit solution is $\varphi^*(x,t) = B^+(x)\phi_U(x,t)$. Substituting the value of $\tilde{G}(x)$, it yields

$$\begin{aligned}\varphi(x,t) &= (G(x)B(x))^{-1}\, G(x)\phi_U(x,t) \\ &= \left(M(x)B^\top(x)B(x)\right)^{-1} M(x)B^\top(x)\phi_U(x,t) \\ &= B^+(x)\phi_U(x,t) = \varphi^*(x,t)\end{aligned}$$

According to Lemma 1, Assumption 3 implies the existence of $\tilde{g}(x)$ generating the Jacobian matrix $\tilde{G}(x)$. Finally, since given a matrix $A \in \mathbb{R}^{n\times m}$, one has (see, e.g. [13]) $||I - A(A^\top A)^{-1}A^\top||_2 = 1$, using $A = B(x)$ one can see that $||I - B(x)(B^\top(x)B(x))^{-1}B^\top(x)||_2 = 1$, $\forall x \in \mathbb{R}^n$, which leads to (20). This implies that it is not possible to obtain an equivalent disturbance with a 2-norm which is smaller than the 2-norm of the unmatched disturbance. ■

B. Case 2: system in regular form

Hereafter, we focus on the task of finding a simple solution for choosing $g(x)$, when the system structure falls into a precise family, as follows.

Assumption 4: System (1) is such that it can be written in the following regular form

$$\dot{x}_{[1]}(t) = f_{[1]}(x,t) + \phi_{[1]}(x,t) \tag{21}$$
$$\dot{x}_{[2]}(t) = f_{[2]}(x,t) + \bar{B}(x)u(x,t) + \phi_{[2]}(x,t) \tag{22}$$

where $x_{[1]} \in \mathbb{R}^{n-m}$, $x_{[2]} \in \mathbb{R}^m$, $[f_{[1]}\ \ f_{[2]}]^\top = f$, $\bar{B} \in \mathbb{R}^{m\times m}$ is a full rank matrix, while $\phi_{[2]} \in \mathbb{R}^m$ and $\phi_{[1]} \in \mathbb{R}^{n-m}$ are the matched and unmatched disturbances, respectively, clearly separable in this form.

Such a structure for the system is often found in the sliding mode control literature, where it is widely used thanks to its nice properties (see, e.g. [1]). By virtue of Assumption 4, for the system that we are considering, it is possible to simply get

$$\begin{aligned}\phi_M(x,t) &= \begin{bmatrix}0 & \dots & 0 & \phi_{[2]_1} & \dots & \phi_{[2]_m}\end{bmatrix}^\top \\ \phi_U(x,t) &= \begin{bmatrix}\phi_{[1]_1} & \dots & \phi_{[1]_{n-m}} & 0 & \dots & 0\end{bmatrix}^\top .\end{aligned}$$

In the following corollary it is shown that, for systems in regular form, it is possible to use a simple linear sliding manifold in the ISM controller design.

Corollary 1: For system (1) fulfilling Assumptions 1 and 4, Problem 1 can be solved by a linear function $\tilde{g}(x) = \tilde{G}x$.

Proof: If Assumption 4 holds, one has

$$B^\perp(x) = \begin{bmatrix}\tilde{B}(x) \\ \mathbf{0}_{m\times(n-m)}\end{bmatrix}$$

where $\tilde{B} \in \mathbb{R}^{(n-m)\times(n-m)}$ is a full rank matrix. If we consider the $i-th$ and $j-th$ columns of $B^\perp(x)$, it yields

$$\frac{\partial B_j^\perp(x)}{\partial x}B_i^\perp(x) - \frac{\partial B_i^\perp(x)}{\partial x}B_j^\perp(x) =$$
$$\begin{bmatrix}\frac{\partial B_{j,1}^\perp}{\partial x_1} & \cdots & \frac{\partial B_{j,1}^\perp}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial B_{j,n-m}^\perp}{\partial x_1} & \cdots & \frac{\partial B_{j,n-m}^\perp}{\partial x_n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}B_{i,1}^\perp \\ \vdots \\ B_{i,n-m}^\perp \\ 0 \\ \vdots \\ 0\end{bmatrix}$$
$$-\begin{bmatrix}\frac{\partial B_{i,1}^\perp}{\partial x_1} & \cdots & \frac{\partial B_{i,1}^\perp}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial B_{i,n-m}^\perp}{\partial x_1} & \cdots & \frac{\partial B_{i,n-m}^\perp}{\partial x_n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}B_{j,1}^\perp \\ \vdots \\ B_{j,n-m}^\perp \\ 0 \\ \vdots \\ 0\end{bmatrix} = \begin{bmatrix}* \\ \vdots \\ * \\ 0 \\ \vdots \\ 0\end{bmatrix} .$$

Since the result of any Lie bracket belongs to the span of the columns of $B^\perp$, the distribution $\Delta(x)$ is involutive, i.e., Assumption 3 is fulfilled (and consequently Assumption 2, which again allows us to refer to Problem 1). Theorem 1 can thus be applied, which leads to the possibility of explicitly finding an integral sliding manifold which minimizes the equivalent disturbance. In particular, the function $\tilde{g}(x)$ can be chosen such that its Jacobian matrix is $\tilde{G}(x) = M(x)B^\top(x)$, choosing $M(x)$ as $M(x) = N\bar{B}^{-\top}(x)$, where $N \in \mathbb{R}^{m\times m}$ is any constant full rank matrix. It yields

$$\tilde{G}(x) = N\bar{B}^{-\top}(x)B^\top(x) = N\bar{B}^{-\top}\begin{bmatrix}0 & \bar{B}^\top\end{bmatrix} = \begin{bmatrix}0 & N\end{bmatrix} = \tilde{G}$$

Therefore, $\tilde{g}(x) = \tilde{G}x = \begin{bmatrix}0 & N\end{bmatrix}x$. ■

Remark 2: This result has a very intuitive meaning: like in the case analyzed in [13] for a constant value of $B(x)$, it is clear that the sliding manifold must be defined such that the ISM control action is not trying to compensate the unmatched disturbance, because any attempt to do it would increase the norm of the equivalent disturbance. In fact, the function $\tilde{g}(x)$ can be expressed as $\tilde{g}(x) = Nx_{[2]}$, which means that the ISM control variable $u_1(t)$ only acts on $\phi_M(x,t)$. This result is possible also in the state-dependent case because the matched disturbance and the control variable act on the same state components at any time instant: as a consequence, the unmatched uncertainties cannot, in any past, present of future time instant, act on the same direction of the matched one and this ensures a "separation" property which makes possible to use a simple sliding manifold to optimize the performances.

IV. A CASE STUDY: DYNAMIC FEEDBACK LINEARIZATION OF THE UNICYCLE

In this section the proposed method will be applied to the unicycle, a very common example of nonholonomic system in mobile robotics (the reader can refer to [19] for an overview on this kind of systems). Without taking the disturbance terms into account, we introduce the kinematic model of the nominal system as

$$\begin{aligned}\dot{x}_1(t) &= u_{v0}(t)\cos x_3(t) \\ \dot{x}_2(t) &= u_{v0}(t)\sin x_3(t) \\ \dot{x}_3(t) &= u_{\omega 0}(t)\end{aligned}$$

where (x_1, x_2) is the position of the robot in Cartesian coordinates in the world reference frame, while x_3 is its orientation with respect to the x_1-axis; u_v and u_ω represent the translational and rotational velocities, respectively, which are regarded as inputs. Note that $u_0 \triangleq [u_{v0}\ u_{\omega 0}]^\top$ is the high level control variable. Therefore, we are considering a third order system with state $x \triangleq [x_1\ x_2\ x_3]$ and two inputs. A possible high level controller for this system was introduced in [20] using dynamic feedback linearization. The structure of the controller is

$$\begin{aligned}\dot{\xi}(t) &= v_1(t)\cos x_3(t) + v_2(t)\sin x_3(t)\\ u_{v0}(t) &= \xi(t)\\ u_{\omega 0}(t) &= \frac{-v_1 \sin x_3(t) + v_2(t)\cos x_3(t)}{\xi(t)}\end{aligned}$$

where $\xi \in \mathbb{R}$ is the state of the dynamic compensator [20]. As for the auxiliary control variables v_1 and v_2, they can be defined according to the following considerations. It is possible to define a new set of coordinates as

$$\begin{aligned}z_1(t) &= x_1(t)\\ z_2(t) &= x_2(t)\\ z_3(t) &= \dot{x}_1(t) = \xi(t)\cos x_3(t)\\ z_4(t) &= \dot{x}_2(t) = \xi(t)\sin x_3(t)\end{aligned}$$

which leads to the possibility of representing the extended system with two chains of integrators $\ddot{z}_1(t) = v_1(t)$, $\ddot{z}_2(t) = v_2(t)$. If the objective is to follow a desired trajectory for $x_1(t)$ and $x_2(t)$, namely $x_{1d}(t)$ and $x_{2d}(t)$, it is possible to design a globally exponentially stabilizing feedback controller defining

$$v_1(t) = \ddot{x}_{1d}(t) + k_{p1}(x_{1d}(t) - x_1(t)) + k_{d1}(\dot{x}_{1d} - \dot{x}_1(t)) \tag{23}$$

$$v_2(t) = \ddot{x}_{2d}(t) + k_{p2}(x_{2d}(t) - x_2(t)) + k_{d2}(\dot{x}_{2d} - \dot{x}_2(t)) \tag{24}$$

with $k_{p1},\ k_{d1},\ k_{p2},\ k_{d2} > 0$. Note that this controller requires that the translational velocity of the robot never goes to zero (see [20] for a detailed analysis of this aspect).

If the presence of disturbances is taken into account, the behavior of the unicycle can be quite different than expected, and ISM can be used to reduce the disturbances. The following system is then considered

$$\begin{aligned}\dot{x}_1(t) &= (u_v(t) + \phi_1(t))\cos x_3(t) - \phi_2(t)\sin x_3(t)\\ \dot{x}_2(t) &= (u_v(t) + \phi_1(t))\sin x_3(t) + \phi_2(t)\cos x_3(t)\\ \dot{x}_3(t) &= u_\omega(t) + \phi_3(t)\end{aligned}$$

where

$$\phi(x,t) \triangleq \begin{bmatrix} \phi_1(t)\cos x_3(t) - \phi_2(t)\sin x_3(t)\\ \phi_1(t)\sin x_3(t) + \phi_2(t)\cos x_3(t)\\ \phi_3(t)\end{bmatrix}$$

is the disturbance vector. If we assume that each component of the vector ϕ is in absolute value smaller than a constant ($\bar{\phi}_1$, $\bar{\phi}_2$, $\bar{\phi}_3$, respectively), we obtain $\Phi = \sqrt{\bar{\phi}_1^2 + \bar{\phi}_2^2 + \bar{\phi}_3^2}$ (as required in Assumption 1), while the control variables are defined as

$$\begin{aligned}u_v(t) &= u_{v0}(t) + u_{v1}(t)\\ u_\omega(t) &= u_{\omega 0}(t) + u_{\omega 1}(t)\end{aligned}$$

u_{v1} and $u_{\omega 1}$ being the ISM contributions. Note that this system can be written in form (1), with $f(x,t) = 0$. To apply the ISM control strategy, we must check if Assumption 3 is fulfilled. The distribution $\Delta(x)$ is

$$\Delta(x) = \text{span}\left\{\begin{bmatrix} -\sin x_3\\ \cos x_3\\ 0\end{bmatrix}\right\}$$

which is involutive since it is spanned by a single vector field. As a consequence, all the assumptions are fulfilled, and the minimization of the disturbance terms can be performed. To define the sliding manifold (3), take

$$g(x) = \begin{bmatrix} x_3\\ x_1\cos x_3 + x_2 \sin x_3\end{bmatrix}$$

with the corresponding

$$G(x) = \begin{bmatrix} 0 & 0 & 1\\ \cos x_3 & \sin x_3 & -x_1\sin x_3 + x_2\cos x_3\end{bmatrix}.$$

Note that, as expected from Lemma 1, this latter can be written as

$$G(x) = \underbrace{\begin{bmatrix} 1 & 0\\ -x_1\sin x_3 + x_2\cos x_3 & 1\end{bmatrix}}_{M(x)}\underbrace{\begin{bmatrix} 0 & 0 & 1\\ \cos x_3 & \sin x_3 & 0\end{bmatrix}}_{B^\top(x)}$$

where $M(x)$ is full rank for all x. The ISM control variable is then computed according to (8). As for the disturbance reduction, exploiting the definition of the unmatched disturbance in (11) (being $B^\perp = [-\sin x_3\ \ \cos x_3\ \ 0]^\top$), it can be computed as

$$\phi_U = \begin{bmatrix} -\phi_2 \sin x_3\\ \phi_2\cos x_3\\ 0\end{bmatrix}$$

leading to $||\phi_U(x,t)||_2 = \phi_2(t)$ and therefore $||\phi_U||_2 \leq \bar{\phi}_2$. The 2-norm of the equivalent disturbance in (14) is obtained as $||\phi_{eq}(x)||_2 = \phi_2(t)$. As expected, the maximum norm of the equivalent disturbance coincides with that of the unmatched disturbance. In conclusion, the disturbance term is reduced to $\Phi = \bar{\phi}_2$, that is the strongest disturbance reduction obtainable with ISM.

In the simulation example shown in the following, the disturbances are chosen as $\phi_1(t) = 1.2\sin(5t)$, $\phi_2(t) = 0.4\sin(20t)$, $\phi_3(t) = 0.8\sin(t)$, leading to $\Phi \simeq 1.5$. The high level controller in (23)-(24) is designed with $k_{p1} = k_{p2} = 15$, $k_{d1} = k_{d2} = 1$, while the ISM control law (8) is defined with a constant gain value $\rho = \sqrt{\bar{\phi}_1^2 + \bar{\phi}_3^2} \simeq 1.45$, in order to compensate the matched disturbance $\phi_M = [\phi_1\cos x_3\ \ \phi_1 \sin x_3\ \ \phi_3]^\top$. Moreover, in order to reduce the so-called chattering effect, the well known equivalent control method [1] is used, applying a linear low-pass filter to the obtained discontinuous control variable. First of all, we show (Fig. 1, top) the path of the unicycle in the $x-y$ plane in case there is no disturbance and the high level controller only is used. As expected, after a transient (since the initial condition is taken on purpose different from the reference), the unicycle trajectory (solid line) settles on the desired one (dashed line). If the disturbances are added, the high level controller has a poor performance (Fig. 1, middle), since it is not designed to work in their presence. Using the proposed ISM strategy, the bound on the disturbances is reduced to $\Phi = 0.4$, and the performance of the overall control law is improving (Fig. 1, bottom). In this last case, we show also (Fig. 2) the time evolution of the control variables u_v and u_ω, and the two components of the sliding manifold s, namely s_1 and s_2. For the reader's interest, a simulation example for systems in regular form can be found in [15].

V. Conclusions

This paper introduces the definition of an integral sliding manifold for general affine nonlinear systems. Two cases are considered, for the general case it is shown that a solution for the minimization of the disturbance terms (i.e., the matched disturbances are eliminated and the unmatched ones are not amplified) can be obtained if some involutivity properties of the system are fulfilled. For systems in regular form, a linear sliding surface can be exploited, obtaining analogous results. The proposed ISM control law is finally tested on a simulation example of a simple nonholonomic system.

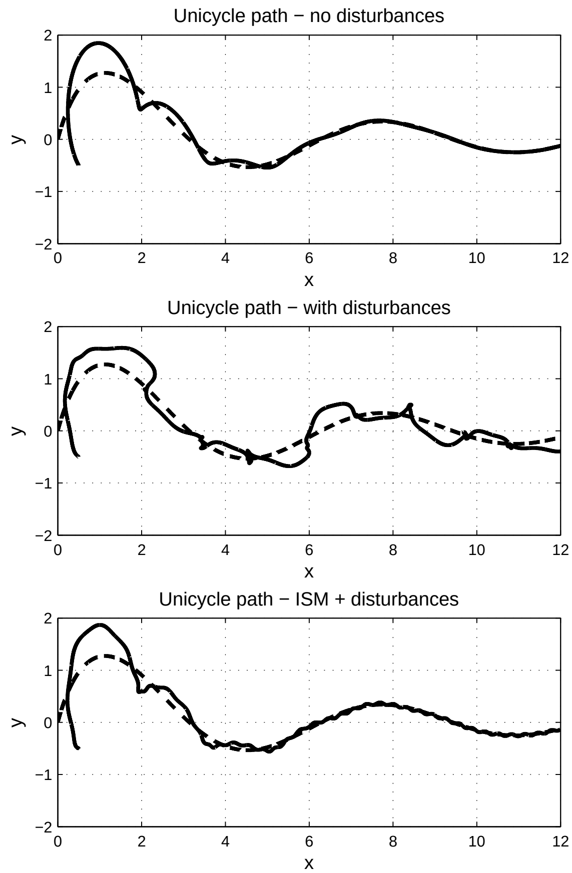

Fig. 1. The path of the unicycle in case of high level controller with no disturbances (top), high level controller with disturbances (middle) and high level controller plus ISM with disturbances (bottom). The reference trajectory is depicted as a dashed line, while the actual ones are represented as solid lines

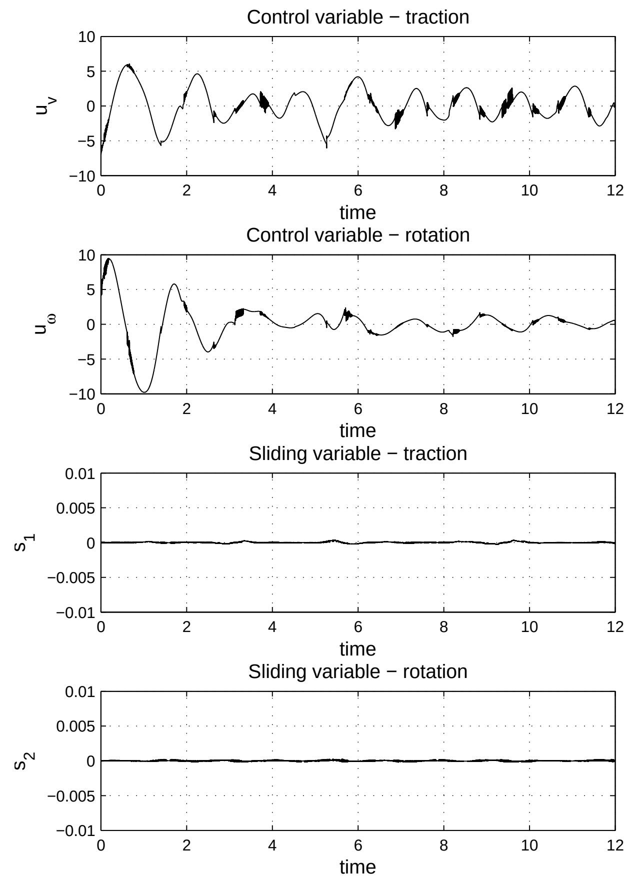

Fig. 2. Time evolution of (from top to bottom) of the control variables and the components of the sliding manifold, for the simulation example regarding the use of high level controller and ISM with disturbances

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