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Control of driftless systems using piecewise constant inputs

Dmitry Gromov[1†] and Fernando Castaños[2]

[1] Faculty of Applied Mathematics and Control Processes, St. Petersburg State University, St. Petersburg, Russia
(Email: dv.gromov@gmail.com)
[2]Automatic Control Department, Cinvestav, México (Email: castanos@ieee.org)

Abstract: In this paper we use geometric tools to establish controllability properties of driftless systems which have less control inputs than states, but whose input vector fields span a non-involutive distribution. Our prototypical class of systems is conformed by kinematic models of non-holonomic systems such as the unicycle or a car with N trailers. We restrict our class of inputs to those which are piecewise constant. The restriction gives way to an easy implementation in discrete time and allows to formulate control problems as systems of polynomial equations. The control problems can then be addressed using geometric-algebraic tools and can be solved explicitly using symbolic computational software if their size is reasonable.

Keywords: Nonholonomic systems, algebraic methods, geometric methods, discrete time control, symbolic computation

1. INTRODUCTION

Controllability of nonlinear systems can be approached from several angles. From the geometric perspective, controllability is assessed by analyzing the distributions that are produced from the Lie brackets of the various vector fields that affect the system under study [6, 8]. A result that has had a large amount of success in this context is Chow's theorem, which provides a necessary and sufficient condition for controllability of driftless systems. Roughly speaking, the condition is that the Lie algebra evaluated at each state must span all of the tangent space of the manifold on which the system evolves. The result has found many applications, e.g., in the control of kinematic models for mobile robots [7].

Geometric tools become particularly relevant for applications when the Lie algebra generated by the vector fields is nilpotent, as this property renders the controllability problem mathematically tractable. One of the simplest examples of a driftless system which has a nilpotent Lie algebra and at the same time satisfies the condition of Chow's theorem is Brockett's celebrated non-holonomic integrator (which happens to be diffeomorphic to the unicycle [5]). In this paper we take interest in such class of systems (nilpotent and which span the whole tangent space). For concreteness, we focus on a five-dimensional extension of the non-holonomic integrator.

In the geometric setting, controllability is understood as the ability to steer the system state from an initial condition to any sufficiently close desired final state. It is well known that, unlike the linear case, controllability does not necessarily imply the possibility of enforcing a desired dynamical behavior. In other words, controllability only allows to conclude the existence of an open loop control law that drives the system state from one point to another. Indeed, results in this area are usually gathered under the name of 'motion planning' and make no reference to important dynamical properties such as stability.

Motion planning has been mostly achieved by the use of sinusoidal inputs whose amplitude and frequencies are carefully chosen in order to achieve the desired motion [1, 7]. Our approach is similar, but the inputs that we consider are piecewise constant. This setting has several advantages: first, the resulting control laws can be directly implemented in discrete time (this can be useful, e.g., when a communication network forbids fast sample times). Second, due to the bilinearity of the Lie brackets, the control problem is translated to a system of polynomial equations that can be dealt with modern algebraic tools already implemented in software packages such as Maple or Maxima. Third, our goal is not only to achieve a final state, but to enforce a desired discrete-time map, so constraints on dynamical properties such as stability can be easily formulated.

We define the notation and state the problem formally in the following section. Section III presents some facts about systems of polynomial equations. Section IV depicts the general methodology and states the main results. Section V contains conclusions and future work.

2. PROBLEM STATEMENT

Let $\mathcal{M}$ be an n-dimensional regular smooth manifold and let $T\mathcal{M}$ be the tangent bundle of $\mathcal{M}$. Consider m linearly independent vector fields $\mathbf{X} = \{\mathbf{X_1}, \ldots, \mathbf{X_m}\}$, $m < n$, $X_i \in T\mathcal{M}$. The set $\mathbf{X}$ spans a linear subset of $T\mathcal{M}$, referred to as the m-dimensional *tangent distribution*, $D_X \subset T\mathcal{M}$. We are interested in the case where the distribution is not involutive. To clarify the point, we first define $\ell(\mathbf{X})$ to be the Lie algebra generated by $\mathbf{X}$, i.e., the union of $\mathbf{X}$ and the set of all Lie brackets $[X_i, V]$, $i = 1, \ldots, m$, $V \in \ell(\mathbf{X})$. A distribution D_X is said to be *non-involutive* if the Lie algebra $\ell(\mathbf{X})$ spans $T\mathcal{M}$. Lie algebra $\ell(\mathbf{X})$ generates a local Lie group $L(\mathbf{X})$ which acts smoothly on the underlying manifold $\mathcal{M}$ with left actions given by exponential maps, $\exp(tZ) : \mathcal{M} \to \mathcal{M}$, $Z \in \ell(\mathbf{X})$. Finally, for each Lie algebra we can define a basis consisting of (a possibly infinite number of) linearly independent Lie brackets, referred to as the *P. Hall basis*, [10, Chap. 11].

† Dmitry Gromov is the presenter of this paper.

The above formalism turns out to be useful when dealing with driftless non-holonomic control systems such that X_i are the control vector fields. The dynamics of such a system is described by a set of differential equations

$$\dot{y}(t) = \sum_{i=1}^{m} u_i(t) X_i(y(t)), \tag{1}$$

where $u_i(t)$ are the piecewise smooth functions, referred to as the controls.

System (1) is said to be *small-time locally controllable* (STLC) at $p \in \mathcal{M}$ if for any sufficiently small $T > 0$ the reachable set from p in time T contains p in its interior. This is equivalent to saying that $L(\mathbf{X})$ acts transitively in a small neighborhood of p, denoted $U(p) \subset \mathcal{M}$, i.e., that the orbit of p is all of $U(p)$. The STLC property for a driftless system (1) can be easily checked thanks to the following theorem (see, e.g., [12, Corollary 7.2]).

Theorem 1 (Chow) System (1) is STLC at $p \in \mathcal{M}$ iff the Lie algebra rank condition (LARC) holds, i.e., $\operatorname{span}\left(\ell(\mathbf{X})\big|_{\mathbf{p}}\right) = \mathbf{T_p}\mathcal{M}$.

In the following we will restrict our attention to the case of a nilpotent Lie algebra $\ell(\mathbf{X})$.

Definition 1: Let the sequence $\mathcal{V}^i$ be defined such that $\mathcal{V}^{i+1} = [\mathbf{X}, \mathcal{V}^{\mathbf{i}}]$ and $\mathcal{V}^0 = \mathbf{X}$. A Lie algebra $\ell(\mathbf{X})$ generated by the vector fields $\mathbf{X}$ is said to be *nilpotent* or order $\gamma \geq 1$ if $\mathcal{V}^\gamma = 0$.

Obviously, $\mathcal{V}^\gamma = 0$ implies $\mathcal{V}^i = 0$ for all $i > \gamma$.

The controllability property implies that for any $p \in \mathcal{M}$ and $U(p) \subset \mathcal{M}$ such that the LARC holds for any $q \in U(p)$ there exists an element g of the group $L(\mathbf{X})$ such that $g(p) = q$. Such an element can be written as an exponential mapping $g = \exp(\tau X_d) : \mathcal{M} \to \mathcal{M}$, $X_d \in \operatorname{span}\left(\ell(\mathbf{X})\right)$. There is however an obstacle in using this fact for control design. The vector field X_d will typically be composed not only of the vector fields from $\mathbf{X}$, but also from their Lie brackets which are not available to us.

The idea is to consider the composition of the exponential mappings involving only the vector fields from $\mathbf{X}$:

$$e^{u_m^k \tau X_m} \cdots e^{u_1^k \tau X_1} \cdots e^{u_m^1 \tau X_m} \cdots e^{u_2^1 \tau X_2} e^{u_1^1 \tau X_1}(p). \tag{2}$$

where u_i^j, $j = 1, \ldots, k$, $i = 1, \ldots, m$ are constants to be defined. This particular ordering might seem restrictive at first (it is not possible, e.g., to construct a symmetric method). In the following section we will show, however, that it is general enough in most cases. This composition corresponds to the solution of the system

$$\dot{y} = u_i(t) X_i(y), \quad y(0) = p$$

with controls

$$u_i(t) = \begin{cases} u_i^j \in \mathbb{R}, & t \in [m\tau j + \tau i, m\tau j + \tau(i+1)), \\ & \quad j = 0, \ldots, k-1 \\ 0, & \text{otherwise}. \end{cases}$$

We assume that all τ's sum up to T, which is the total length of the interval. In other words, we split the time interval $[0, T]$ into km subintervals $\tau = \frac{T}{km}$ and "actuate" the i-th vector field with the constant factor (control) u_i^1 during a single interval. As we have gone through all vector fields, the procedure repeats with controls u_i^2 and so on k times. Note that the piecewise control can be readily implemented in discrete-time.

The composition (2) can be represented as the exponential mapping of a new vector field X_u. To do so we will make use of the Baker-Campbell-Hausdorff (BCH) formula, which for the case of two exponential mappings and keeping only the Lie brackets up to the third order is

$$Z(Y, X) = \log(\exp Y \exp X) = X + Y - \frac{1}{2}[X, Y]$$
$$+ \frac{1}{12}\left([Y, [Y, X]] + [X, [X, Y]]\right) + \frac{1}{24}[Y, [X, [X, Y]]] + \cdots$$

One can observe that the following identity holds:

$$Z(X, Y) = -Z(-Y, -X).$$

The resulting vector field can be written as a composition of the basis vectors of $\ell(\mathbf{X})$: $X_c = \sum_{i=1}^{h} v_i(u) V_i$, $h = \dim(\ell(\mathbf{X}))$, $V_i \in \ell(\mathbf{X})$, and $v_i(u)$ are polynomials in u_i^j. This can be interpreted as follows: the composition of elementary movements generated by individual vector fields results in a system evolution that can be considered as being produced by a new vector field.

Using this approach one can formulate the following constructive algorithm for determining the control inputs. Given a point $p \in \mathcal{M}$,

1. Determine the desired vector field $X_d(p)$.
2. Write $X_d(p)$ in terms of the Lie algebra basis:

$$X_d(p) = \sum_{i=1}^{h} d_i V_i(p),$$

$V_i(p) \in \ell(\mathbf{X})\big|_{\mathbf{p}}$, where d_i are constants.

3. Fix k and write the vector field $X_c(p)$ resulting from the composition of k sets of m vector fields $X_i(p)$, again in terms of the Lie algebra basis:

$$X_c(p) = \sum_{i=1}^{h} v_i(u) V_i(p),$$

where $v_i(u)$ are polynomial equations in variables u_i^j of degree not exceeding γ, with γ being the order of nilpotency of the respective Lie algebra.

4. Determine the controls u_i^j from solving the system of polynomial equations $v_i(u) = d_i$.

The outlined procedure can be repeated at each discrete time instant $t = jT$, $j = 0, 1, \ldots$, thus yielding a discrete-time feedback control.

3. SYSTEMS OF POLYNOMIAL EQUATIONS

In this section we will present some basic facts related to solving (systems of) polynomial equations. For a detailed exposition on the subject see [2, Chap. 3], and [11, Chap. 4].

We first recall some basic facts about solving polynomial equations. Let $x = \{x_i\}$, $i = 1, \ldots, n$, be a set of variables. A *monomial* is a product $\sigma(x) = \prod_{j=1}^{n} x_j^{k_j}$, where $k_j \in \mathbb{N} \cup \{0\}$ are the non-negative integer powers. The sum of powers $k = \sum_{j=1}^{n} k_j$ is said to be the *degree* of the monomial. A multivariate polynomial $p(x)$ is a sum of monomials in x multiplied by coefficients, $p(x) = \sum_{i=1}^{l} a_i \sigma_i(x)$, where $\sigma_i(x)$ are the respective monomials and the coefficients are assumed to be from a field $\mathbb{F}$, $a_i \in \mathbb{F}$. In the context of the problem under consideration, $\mathbb{F}$ is the set of either rational or real numbers. A polynomial is said to be univariate (resp. bivariate) if the set of variables x consists of one (resp., two) elements. The degree of the polynomial $p(s)$, denoted $\deg(p)$, is defined as the maximum of the degrees of monomials contained in the polynomial. A polynomial is said to be *homogeneous* of order m if all its monomials have the same degree m. The notion of homogeneity becomes trivial for univariate polynomials.

Definition 2: Consider two univariate polynomial equations,

$$p(x) = \sum_{i=0}^{m} a_i x^{m-i} = 0, \quad g(x) = \sum_{j=0}^{l} b_j x^{l-j} = 0, \quad (3)$$

the *resultant* of $p(x)$ and $g(x)$, denoted $\mathcal{R}(p, g)$, [2, §1, Chap. 3], is a polynomial function of the coefficients a_i and b_j which is defined as the determinant of the Sylvester matrix associated with $p(x)$ and $g(x)$.

The resultant has a particularly important property as stated in the following theorem (see [3, §6, Chap.3]).

Theorem 2: Polynomials $p(x)$ and $g(x)$ have a common root if and only if their resultant is equal to zero, i.e., $\mathcal{R}(p, g) = 0$.

Note that the above result is equivalent to saying that $p(x)$ and $g(x)$ have a common factor $(x - \bar{x})$, where $\bar{x}$ is their common root.

In particular, the resultant of $p(x)$ and $g(x)$ can be represented in the following form:

$$\mathcal{R}(p, g) = a_0^m b_0^l \prod_{i=1}^{m} \prod_{j=1}^{l} (\lambda_i - \mu_j),$$

where λ_i and μ_j are the roots of $p(x)$ and $g(x)$. The resultant of a univariate polynomial $p(x)$ and its first derivative w.r.t. x is called the *discriminant*, $\mathcal{D}(p(x)) = \mathcal{R}(p(x), p'(x))$. Theorem 2 implies that the discriminant of $p(x)$ is equal to zero when $p(x)$ has a root of multiplicity greater than 1.

Resultants are very useful in solving systems of polynomial equations. Consider the system of two bivariate polynomials in (x, y):

$$p(x, y) = 0, \quad g(x, y) = 0. \quad (4)$$

Each of these polynomials can be written as a polynomial in one variable with coefficients expressed as polynomials in the remaining variable. Say, $p(x, y)$ can be written as

$$p(x, y) = \sum_{i=0}^{m} a_i(y) x^{m-i}$$

or, alternatively as

$$p(x, y) = \sum_{i=0}^{\bar{m}} \bar{a}_i(x) y^{\bar{m}-i},$$

where, m and $\bar{m}$ are the largest exponents of x and y (in general, $m \neq \bar{m}$) and $\deg(a_0) \leq \deg(p) - m$, resp., $\deg(\bar{a}_0) \leq \deg(p) - \bar{m}$. Let $p(x, y)$ and $g(x, y)$ be represented as above with $b_i(y)$ and $\bar{b}_i(x)$ being the coefficients of $g(x, y)$. In the following, we will require the following regularity assumption to hold.

Assumption 3: Coefficients $a_0(y)$, $\bar{a}_0(x)$ and $b_0(y)$, $\bar{b}_0(x)$ are all non-zero.

We can define resultants of polynomials (4) w.r.t. one of the variables assuming the second one to be fixed:

$$\begin{aligned} \mathcal{X}(x) &= \mathcal{R}_y(p(x, y), g(x, y)), \\ \mathcal{Y}(y) &= \mathcal{R}_x(p(x, y), g(x, y)). \end{aligned} \quad (5)$$

A solution of (4) is characterized by the following theorem (cf. [4, Thm. 1.4.1]).

Theorem 3: If Assumption 3 holds and the pair (a, b) solves (4), then $\mathcal{X}(a) = 0$ and $\mathcal{Y}(b) = 0$.

Thus, solving (4) reduces to solving a polynomial in one variable, either $\mathcal{X}(x) = 0$ or $\mathcal{Y}(y) = 0$. Let a be a root of $\mathcal{X}(x)$, then there exists such $y = b$ that $\mathcal{R}_y(p(a, y), g(a, y)) = 0$ holds and hence the system $p(a, y) = 0$, $g(a, y) = 0$ has a common root according to Thm. 2.

The procedure of solving a system of polynomial equations by subsequent elimination of variables is referred to as *elimination theory*. The following variation of the *Shape lemma* (see [9]) shows a constructive way of computing a solution to (4).

Theorem 4: If Assumption 3 holds and $\mathcal{D}(\mathcal{X}(x)) \neq 0$ (i.e., $\mathcal{X}(x)$ does not have multiple roots) then the system (4) is equivalent to the following system:

$$\mathcal{X}(x) = 0, \quad y = F(x),$$

where $F(x)$ is a rational function over $\mathbb{F}$.

Theorem 4 effectively says that if there exists a real, multiplicity 1 root of $\mathcal{X}(x) = 0$, then the remaining variable y is also real and can be efficiently computed as $y = F(x)$.

The elimination procedure described above can equally well be applied to solving systems of polynomial equations in more than 2 variables. The procedure will be briefly outlined for a system of three equations in three variables:

$$p(x, y, z) = 0, \quad g(x, y, z) = 0, \quad f(x, y, z) = 0.$$

The most straightforward solution is to eliminate z by computing resultants

$$\begin{aligned} \mathcal{X}(x, y) &= \mathcal{R}_z(p(x, y, z), g(x, y, z)) \\ \mathcal{Y}(x, y) &= \mathcal{R}_z(g(x, y, z), f(x, y, z)) \end{aligned} \quad (6)$$

and obtain the ultimate resultant (also called *eliminant*) by computing the resultant of $\mathcal{X}(x,y)$ and $\mathcal{Y}(x,y)$,

$$\mathcal{Z}(x) = \mathcal{R}_y(\mathcal{X}(x,y), \mathcal{Y}(x,y))$$

and follow the lines described above. However, it turns out that the resulting eliminant may also yield the so called *extraneous solutions*, that is, roots that do not correspond to any solution of the original system. This problem can be resolved by computing all cross-resultants and extracting their greatest common divisor, which represents the sought for the eliminant.

Typically, the roots of a polynomial belong to the algebraically closed field extension of $\mathbb{F}$, which is $\mathbb{C}$ for $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{R}$. However, in most applications one is interested in determining the real roots as these correspond to physically realizable solutions. So far we assumed that such a real-valued solution exists. Furthermore, Theorem 4 states that, under certain non-restrictive conditions, the existence of a real root of the eliminant polynomial implies that the remaining variables will also assume real values. Thus, one can guarantee existence of at least one real solution. In many applications this is sufficient.

We conclude by considering one particular result that will be useful in the following. Let $p(x,y)$ be such that $p(x,y) = p(x,-y)$. This implies that one can make a change of variables $Y = y^2$ to get $P(x,Y) = p(x,y)$. We have the following theorem [13].

Theorem 5: Let $p(x,y) = p(x,-y)$. Let furthermore the following conditions hold simultaneously:
1. All solutions of the equation $p(x,0) = 0$ are complex, and
2. Equation $\mathcal{P}(z) = \mathcal{D}_x(P(x, z - x^2)) = 0$ does not have positive solutions.
Then the equation $p(x,y) = 0$ does not have a real valued solution.
If either of these conditions does not hold, the real root is either given as a solution to $p(x,0) = 0$, or it is computed as a solution to the system

$$\begin{aligned} p(x,y) &= 0 \\ x^2 + y^2 &= \sqrt{\zeta}, \end{aligned}$$

where ζ is the least positive root of $\mathcal{P}(z) = 0$.

4. DETAILING OF THE METHOD

To illustrate the proposed approach we consider a nonholonomic control system with control vector fields X and Y. The Lie algebra $\ell(X,Y)$ generated by $\{X,Y\}$ is assumed to be nilpotent of order 3. This implies that all iterated Lie brackets of order 3 and higher are equal to 0. The Lie algebra $\ell(X,Y)$ is therefore finite-dimensional and its basis consists of the following elements (written according to the P. Hall convention): X, Y, $[X,Y]$, $[X,[X,Y]]$, and $[Y,[Y,X]]$.

Example 4: Consider the vector fields

$$X = \frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_3} + x_2^2\frac{\partial}{\partial x_5}$$

and

$$Y = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_1^2\frac{\partial}{\partial x_4}\ .$$

X and Y generate a nilpotent Lie algebra of order 3 that satisfies

$$\begin{aligned} [X,Y] &= 2\frac{\partial}{\partial x_3} + 2x_1\frac{\partial}{\partial x_4} - 2x_2\frac{\partial}{\partial x_5}, \\ [X,[X,Y]] &= 2\frac{\partial}{\partial x_4}, \\ [Y,[Y,X]] &= 2\frac{\partial}{\partial x_5}, \\ [X,[X,[X,Y]]] &= 0, \\ [Y,[X,[X,Y]]] &= 0. \end{aligned}$$

It is worth noting that this pair of vector fields serve as a model for the ball-plate problem [1].

We will consider the composition of the exponential mappings

$$e^{v_m\tau Y}e^{u_m\tau X}\cdots e^{v_1\tau Y}e^{u_1\tau X}(p), \qquad (7)$$

where u_i, and v_i, $i = 1,\ldots,m$ are constants to be defined. Since the basis of $\ell(X,Y)$ is five-dimensional, it suffices to set $m = 3$. Thus, there will be 6 constants in total.

The computation of (7) can be automated by using the approach described above. Consider two vector fields

$$\begin{aligned} W &= a_1X + a_2Y + a_3[X,Y] + a_4[X,[X,Y]] + a_5[Y,[Y,X]] \\ Z &= b_1X + b_2Y + b_3[X,Y] + b_4[X,[X,Y]] + b_5[Y,[Y,X]], \end{aligned}$$

where a_i and b_i are real valued constant coefficients. The composition of the respective exponential mappings can be written using the BCH formula:

$$\begin{aligned} &\log\big[\exp(Z)\exp(W)\big] = \\ &c_1X + c_2Y + c_3[X,Y] + c_4[X,[X,Y]] + c_5[Y,[Y,X]], \end{aligned}$$

where the coefficients c_i are functions of a_i and b_i:

$$\begin{aligned} c_1 &= a_1 + b_1 \\ c_2 &= a_2 + b_2 \\ c_3 &= a_3 + b_3 + \frac{1}{2}(a_2b_1 - a_1b_2) \\ c_4 &= a_4 + b_4 + \frac{1}{2}(a_3b_1 - a_1b_3) + \frac{1}{12}(a_1 - b_1)(a_1b_2 - b_1a_2) \\ c_5 &= a_5 + b_5 - \frac{1}{2}(a_3b_2 - a_2b_3) + \frac{1}{12}(b_2 - a_2)(b_2a_1 - b_1a_2). \end{aligned} \qquad (8)$$

Applying (8) iteratively we get the following expression for (7):

$$\begin{aligned} &\log\big[e^{v_3\tau Y}e^{u_3\tau X}e^{v_2\tau Y}e^{u_2\tau X}e^{v_1\tau Y}e^{u_1\tau X}\big] = \\ &\delta_1\tau X + \delta_2\tau Y + \delta_3\tau^2[X,Y] \\ &\qquad + \delta_4\tau^3[X,[X,Y]] + \delta_5\tau^3[Y,[Y,X]], \end{aligned} \qquad (9)$$

where we have set $m = 3$ and

$$\delta_1 = u_1 + u_2 + u_3$$
$$\delta_2 = v_1 + v_2 + v_3$$
$$\delta_3 = \frac{1}{2}(v_1u_2 - u_1v_1 - v_2u_1 - v_2u_2 + u_3v_1 + u_3v_2 - v_3u_1 - v_3u_2 - v_3u_3)$$
$$\delta_4 = \frac{1}{12}\big[(u_1^2 + u_2^2 + u_3^2)(v_1 + v_2 + v_3) - 4u_1u_2v_1 + 2v_2u_1u_2 - 4u_3u_1v_1 - 4u_3v_2u_1 + 2u_3v_1u_2 - 4u_3v_2u_2 + 2v_3u_1u_2 + 2v_3u_1u_3 + 2v_3u_2u_3\big]$$
$$\delta_5 = \frac{1}{12}\big[(u_1 + u_2 + u_3)(v_1^2 + v_2^2 + v_3^2) + 2v_1v_2u_1 - 4v_1v_2u_2 + 2u_3v_1v_2 + 2v_3u_1v_1 - 4v_3v_1u_2 - 4v_3u_3v_1 + 2v_3v_2u_1 + 2v_3v_2u_2 - 4v_3u_3v_2\big]. \quad (10)$$

The constants δ_i can be interpreted as coefficients of the desired vector field X_d. Thus, the following general control problem can be formulated.

Problem 1: Given a vector field (9) with arbitrarily chosen coefficients δ_i. Find at least one set of real-valued solutions $\{u_i, v_i\}$, $i = 1, \ldots, 3$ of (10).

To simplify our further analysis we assume that $\delta_1 \neq 0$ and $\delta_2 \neq 0$. The case where either or both of these coefficients are equal to 0 can be considered along the same line with minor modifications[1]. Using this assumption we can modify (10) to reduce the number of parameters δ_i. To do so we introduce new controls $\bar{u}_i = u_i/\delta_1$, $\bar{v}_i = v_i/\delta_2$. Performing the substitution and dividing the resulted polynomial equations by δ_1, δ_2, $\delta_1\delta_2$, $\delta_1^2\delta_2$, and $\delta_1\delta_2^2$ we arrive at

$$1 = \bar{u}_1 + \bar{u}_2 + \bar{u}_3$$
$$1 = \bar{v}_1 + \bar{v}_2 + \bar{v}_3$$
$$d_3 = \frac{1}{2}(\bar{v}_1\bar{u}_2 - \bar{u}_1\bar{v}_1 - \bar{v}_2\bar{u}_1 - \bar{v}_2\bar{u}_2 + \bar{u}_3\bar{v}_1 + \bar{u}_3\bar{v}_2 - \bar{v}_3\bar{u}_1 - \bar{v}_3\bar{u}_2 - \bar{v}_3\bar{u}_3)$$
$$d_4 = \frac{1}{12}\big[(\bar{u}_1^2 + \bar{u}_2^2 + \bar{u}_3^2)(\bar{v}_1 + \bar{v}_2 + \bar{v}_3) - 4\bar{u}_1\bar{u}_2\bar{v}_1 + 2\bar{v}_2\bar{u}_1\bar{u}_2 - 4\bar{u}_3\bar{u}_1\bar{v}_1 - 4\bar{u}_3\bar{v}_2\bar{u}_1 + 2\bar{u}_3\bar{v}_1\bar{u}_2 - 4\bar{u}_3\bar{v}_2\bar{u}_2 + 2\bar{v}_3\bar{u}_1\bar{u}_2 + 2\bar{v}_3\bar{u}_1\bar{u}_3 + 2\bar{v}_3\bar{u}_2\bar{u}_3\big]$$
$$d_5 = \frac{1}{12}\big[(\bar{u}_1 + \bar{u}_2 + \bar{u}_3)(\bar{v}_1^2 + \bar{v}_2^2 + \bar{v}_3^2) + 2\bar{v}_1\bar{v}_2\bar{u}_1 - 4\bar{v}_1\bar{v}_2\bar{u}_2 + 2\bar{u}_3\bar{v}_1\bar{v}_2 + 2\bar{v}_3\bar{u}_1\bar{v}_1 - 4\bar{v}_3\bar{v}_1\bar{u}_2 - 4\bar{v}_3\bar{u}_3\bar{v}_1 + 2\bar{v}_3\bar{v}_2\bar{u}_1 + 2\bar{v}_3\bar{v}_2\bar{u}_2 - 4\bar{v}_3\bar{u}_3\bar{v}_2\big], \quad (11)$$

where $d_3 = \frac{\delta_3}{\delta_1\delta_2}$, $d_4 = \frac{\delta_4}{\delta_1^2\delta_2}$, and $d_5 = \frac{\delta_5}{\delta_1\delta_2^2}$ (recall that $\delta_1 \neq 0$ and $\delta_2 \neq 0$). In the following, we will work with (11) dropping the bars over u_i and v_i.

Now we are ready to formulate the first result.

[1] Actually, those cases are even easier to analyze as we have at least one homogeneous equation which is typically easier to deal with.

Theorem 6: System (11) has a solution for any choice of parameters (d_3, d_4, d_5).

Proof: The proof is based on computing the resultant of the system of polynomials (11), which (after rescaling and removing extraneous factors) takes the form

$$\mathcal{R}(u_3, v_3) = 2\big[(6d_3 + 3)v_3^2 + (12d_5 - 6d_3 - 4)v_3 - 12d_5 + 1\big]u_3^2 + \big[(24d_4 + 12d_3 + 4)v_3^2 + (36d_3^2 - 48d_4 + 12d_3 + 1)v_3 + \alpha\big]u_3 - \big[24d_3d_5 + 24d_4d_3 + 8d_5 + 48d_4d_5 - d_3 + 8d_4 - 6d_3^2 - 12d_3^3 - \frac{1}{6}\big] \quad (12)$$

with $\alpha = -12d_3^2 - 4d_3 + 24d_4 + 24d_5 + 48d_3d_5 - 1$. The system (11) has a solution if there exist $u_3^* \in \mathbb{R}$ and $v_3^* \in \mathbb{R}$ such that $\mathcal{R}(u_3^*, v_3^*) = 0$. Note that the resultant (12) can be represented as a quadratic equation in terms of either u_3 or v_3 with coefficients depending on the remaining variable. We chose the former as in (12) and write the discriminant of the quadratic equation as

$$D(v_3) = \frac{1}{144}(6d_4 + 3d_3 + 1)^2 v_3^4 + \ldots,$$

where only the high-order term was retained. There are three cases:

- $6d_4 + 3d_3 \neq -1$. The coefficient in front of v_3^4 is positive and hence it is always possible to chose v_3 such that the discriminant $D(v_3) > 0$ which, in turn, implies that there exists at least one real-valued solution to $\mathcal{R}(u_3^*, v_3^*) = 0$.
- $6d_4 + 3d_3 + 1 = 0$, $d_4 \neq 1/12$. The discriminant $D(v_3)$ reduces to a quadratic equation with the coefficient proportional to $(12d_4 - 1)^4$ in front of the leading term. As the coefficient of the leading term is positive we arrive at the same conclusion as in the previous case.
- $6d_4 + 3d_3 + 1 = 0$, $d_4 = 1/12$. In this case the resultant (12) turns into $\mathcal{R}(u_3, v_3) = (v_3 - 12v_3d_5 + 12d_5 - 1)\,u_3^2$, which can be easily solved to yield a pair of real-valued controls (u_3^*, v_3^*).

Finally, having computed (u_3^*, v_3^*), the remaining variables can be computed recursively using a multivariant analogue of Theorem 4. ■

Note that the set of controls u_i^*, v_i^* will in general be non-unique. To single out a particular solution one could employ some additional criteria as described below. But first we present a result that can be easily derived in a way similar to Theorem 6.

Theorem 7: Let $v_3 = 0$. Then the system (11) has a solution for a set of parameters (d_3, d_4, d_5) if the following condition holds:

$$\big(12d_3^2 - 24d_4 - 1\big) \times \big(36d_3^2 - 96d_5 - 72d_4 + 576d_5^2 + 1\big) > 0. \quad (13)$$

Proof: This result can be proven in the same way as Theorem 6, but setting v_3 equal to 0. The resultant $\mathcal{R}(u_3)$ turns out to be a quadratic function in u_3. Equation (13) corresponds to the case when the respective discriminant is positive, thus guaranteeing that the equation

$\mathcal{R}(u_3) = 0$ has exactly two distinct real-valued roots u_3^*. The remaining variables are derived from u_3^*. ■

Note that we required (13) to be positive in order to avoid the situation when there is a real root of multiplicity 2. This case can be considered as well, but requires more attention.

One particularly interesting problem statement that can be addressed within this framework is related to the case where the controls associated to any of two vector fields can assume only positive values. This can be formulated as follows.

Problem 2: Given a vector field (9) with arbitrary chosen coefficients δ_i. Determine the control values $u_i > 0$ and $v_i \in \mathbb{R}$ (resp. $v_i > 0$ and $u_i \in \mathbb{R}$), $i = 1, \ldots, 3$ satisfying system (10).

Problem 2 can be seen as a step toward a general control system with a drift term. Indeed, a system with drift differs from a driftless system in that there is a particular vector field (the drift vector field) that cannot be arbitrarily controlled. In particular, a drift vector field X does not admit a backward solution, i.e., $\exp(-\tau X)$, a situation which may well indeed be possible for a driftless system with a negative control (say, $u_i < 0$ for some i).

Assume for certainty that $u_i > 0$, $i = 1, \ldots, 3$. The following Lemma shows that Problem 2 can be reduced to solving Problem 1.

Lemma 8: Problem 2 has a solution if system (10) with η_i^2 substituted for u_i has a real-valued solution.

Proof: The modified system of polynomial equations will have pairs of roots symmetric w.r.t. the origin. There are two cases: a positive and a negative root correspond to a positive u_i root of (10); two complex-conjugate purely imaginary roots correspond to a negative u_i root of (10). ■

Lemma 8 shows a constructive way to incorporate the restrictions on the sign of the control variable. Note that the resulting system of polynomial equations can be efficiently analyzed for the existence of real roots using Theorem 5.

5. CONCLUSIONS

By considering sequences of piecewise constant controls and exploiting the bilinear property of the Lie bracket, it is possible to formulate many control problems as systems of polynomial equations. We show that, for the five-dimensional extension of the non-holonomic integrator, it is possible to establish any desired dynamics in discrete-time.

It is likely that similar results will hold for other systems which satisfy Chow's theorem and whose corresponding Lie algebra is nilpotent. In any case, concrete systems can be analyzed within the proposed framework using readily available mathematical tools. The framework also allows for easy formulation of control constraints.

As future work we consider the controllability problem for a control system with drift. A general statement such as Chow's theorem will probably be unattainable, but we expect controllability to be easily assessed on a case by case basis. Another potential area of interest is the case of Lie algebras that are not nilpotent.

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