Homogeneous Generalisation of the Lur'e Problem and the Circle Criterion

Emanuel Rocha [*] **Jaime A. Moreno** [**] **Fernando Castaños** [*]

[*] *Departamento de Control Automático, Cinvestav-IPN, 07360 Ciudad de México, México (e-mail: {erocha, fcastanos}@ctrl.cinvestav.mx).*
[**] *Instituto de Ingeniería, Universidad Nacional Autónoma de México, 04510 Ciudad de México, México (e-mail: jmorenop@iingen.unam.mx)*

Abstract. Homogeneous systems are an interesting generalisation of the class of linear systems. On the one hand, many of the properties for which linear systems are useful in the analysis of more complex systems are shared by their homogeneous counterparts. On the other hand, they exhibit a considerably larger set of behaviours. In this article a generalisation of the Lur'e problem of absolute stability is presented as a result of an analysis on homogeneous systems. Moreover, a solution to this problem is introduced as the homogeneous extension of the circle criterion.

Keywords: Absolute stability, Homogeneous systems, Lur'e problem, Circle criterion.

1. INTRODUCTION

To this day, the impossibility of constructing a general theory of non linear systems is clear, which compels us to focus our attention on certain classes of systems and to develop models and control algorithms adapted to each class.

The focus of this article is on the class of homogeneous systems, though the final objective is to model and control systems that need not necessarily be homogeneous. In a sense, we strive to extend some of the accomplishments of the linear control theory with regard to the modelling and control of complex physical systems that might eventually be non linear.

The family of homogeneous systems appears to be an interesting alternative because, on the one hand, it strictly contains the class of linear systems and, on the other hand, it makes possible to analyse behaviours that are truly non linear by using a set of analytic tools that have been under development over the years, see a recent survey by Kawski (2015), and the references therein.

In general, a homogeneous model can approximate a given system with better precision than a linear model (Hermes, 1991). It is convenient to use homogeneous models when, for instance, the linear approximation of the system under analysis does not preserve fundamental properties such as controllability or stabilisability, but they are instead preserved in homogeneous approximations of higher order.

Furthermore, homogeneous systems give rise to more ambitious objectives of control, such as finite-time stabilisation, see Bhat and Bernstein (1997), which is not possible in a purely linear context. Homogeneous controllers, which are developed within the framework of sliding mode control (Levant, 2005), also have robustness properties that make them particularly attractive.

An important result in the theory of linear systems has its roots in the study of the stability of a feedback loop consisting of a nominal linear system in the forward path and a non linear memoryless function in the feedback path, in other words, the analysis of the stability of Lur'e systems (Lur'e, 1944).

When such a system is stable for a whole class of non linear functions, we speak of *absolute stability*. From the day of its statement, the Lur'e problem has been widely studied by the control community because it relates to the analysis of the stability of systems with uncertainties. Moreover, it has a strong connection to the passivity theory and its role in the design of non linear observers, (see, for instance, Brogliato and Heemels (2009)). A solution to the absolute stability problem is given by the circle criterion which, from the state-space representation perspective, is a consequence of the Kalman-Yakubovich-Popov (KYP) lemma, hence its connection to the passivity property of systems.

Absolute stability is, without a doubt, a paramount property that has permeated the control theory literature and whose extension to more general systems is an attractive possibility. For Lur'e-like systems in which the nominal part is nonlinear, the theorem by Hill and Moylan (1976) provides a set of sufficient conditions for absolute stability. However, this result is not a constructive one in the sense that the storage function must be specified *a priori*. Furthermore, it strongly restricts the class of possible storage functions. For example, in the linear case, only quadratic storage functions can be considered.

In this article, we propose a generalisation of the circle criterion different from the one given by the Hill-Moylan theorem. This criterion allows for a larger class of feasible

[⋆] The authors gratefully acknowledge the financial support from CONACyT CVU 627584.

storage functions, making it possible to consider non-quadratic ones in the analysis of Lur'e systems.

On the other hand, we focus on nominal systems and storage functions which are homogeneous, allowing a more constructive approach to the problem of finding the appropriate storage functions. It is possible, in principle, to use tools such as sum of squares (SOS) and Pólya's Theorem, to name a few.

The present article is organized as follows: Section 2 provides the necessary concepts for the rest of the document. Section 3 presents the problem of absolute stability in its classic form, as a reminder, and its generalisation to a class of homogeneous systems. Section 4 contains the main result of the article, which is the solution to the absolute stability problem by using a generalisation of the circle criterion. An example of an application of this result is presented in Section 5 and it is subject to discussion in Section 6. Finally, the conclusions of this work are given in Section 7.

2. PRELIMINARIES

In this section, the concepts of homogeneous function, homogeneous vector field, and homogeneous Lyapunov function are presented. The term "homogeneity" throughout this work refers to the concept of *weighted homogeneity* (Sepulchre and Aeyels, 1996; Bacciotti and Rosier, 2006; Grüne, 2000). Let us start with some basic definitions:

Definition 1. A mapping

$$\delta_\varepsilon^r x = (\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n)^T, \quad \forall \varepsilon > 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}$$

is said to be a *dilation* on $\mathbb{R}^n$, where $r = (r_1, r_2, \ldots, r_n)^T$ and $0 < r_i < \infty$, $i = 1, 2, \ldots, n$.

Definition 2. A function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be a *homogeneous function* of degree $\sigma \in \mathbb{R}$ with respect to $\delta_\varepsilon^r x$, this is denoted as $h \in H_\sigma$, if $h(\delta_\varepsilon^r x) = \varepsilon^\sigma h(x)$.

Definition 3. A system

$$\dot{x} = f(x, u) \tag{1}$$

where $u \in \mathbb{R}$ is the input and $x \in \mathbb{R}^n$ is the state vector, is said to be a *homogeneous system* of degree τ with respect to $\delta_\varepsilon^r x$ and $\varepsilon^s u$ or, equivalently, $f \in \underline{n}_\tau$ if

$$f(\delta_\varepsilon^r x, \varepsilon^s u) = \varepsilon^\tau \delta_\varepsilon^r f(x, u). \tag{2}$$

Notice that a vector field $f(x, u(x)) = \sum_{j=1}^n f_j \frac{\partial}{\partial x_j}$ is homogeneous of degree τ if and only if each component f_j is a homogeneous function of degree $\tau + r_j$ with respect to the dilation $\delta_\varepsilon^r x$. Therefore, in particular, a linear vector field (as in the linear system $\dot{x} = Ax + Bu$) is homogeneous of zero degree with respect to the *standard dilation* (this is $r = (1, 1, \ldots, 1)^T$).

Definition 4. A class C^1 proper positive-definite homogeneous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be *a strict homogeneous Lyapunov function* for the system $\dot{x} = f(x)$ if

$$\nabla V(x) f(x) < 0, \quad x \neq 0, \quad \text{where } \nabla V(x) = \frac{\partial}{\partial x} V(x).$$

Apart from the previous definitions, let us introduce the following stability theorem to which we will refer in a forthcoming section.

Theorem 1. [Rosier (1992)] Let f be a vector field on $\mathbb{R}^n$ such that the origin is a locally asymptotically stable (AS) equilibrium point. Assume that $f \in \underline{n}_\tau$ for some $r \in (0, \infty)^n$. Then, for any positive integer p and any $m > p \cdot \max_i \{r_i\}$, there exists a strict Lyapunov function V for system $\dot{x} = f(x)$, which is δ_ε^r-homogeneous of degree m and of class C^p. As a direct consequence, the time-derivative $\dot{V} = \nabla V f \in H_{m+\tau}$.

3. PROBLEM STATEMENT

The solution to the Lur'e problem of absolute stability has a close relationship with a number of conjectures that either have been refuted by counterexamples or have not yet been confirmed. Nonetheless, these conjectures were the starting point from which the solution was reached. Let us, as a reminder, write Aizerman's and Kalman's, two of the most representative and well-known of these conjectures.

Consider the following feedback loop (Lur'e system).

$$\dot{x} = Ax + Bu, \tag{3}$$

$$y = Cx \tag{4}$$

$$u = -\psi(t, y) \tag{5}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are, respectively, the state vector, the input and the output of the nominal system. Matrices A, B, and C are constant. The feedback interconnection (5) is given by a non-linear memoryless function which satisfies a *sector condition*, this is $\psi \in [k_1, k_2]$, and it is said that ψ belongs to the sector $[k_1, k_2]$, if it satisfies

$$(\psi(t, y) - k_1 y)(\psi(t, y) - k_2 y) \leq 0 \qquad \forall t \in \mathbb{R}_+, \quad \forall y \in \mathbb{R}, \tag{6}$$

where k_1, k_2 are such that $k_2 > k_1$. The sector can also be defined by $[k_1, \infty]$ if

$$y(\psi(t, y) - k_1 y) \geq 0 \quad \forall t \in \mathbb{R}_+, \quad \forall y \in \mathbb{R}. \tag{7}$$

Consider now the case $\psi(t, y) = ky$, $k \in [k_1, k_2]$ in system (3)-(5). That is, ψ is a linear function in the sector $[k_1, k_2]$. Suppose that the resulting system is AS for all $k \in [k_2, k_2]$. Aizerman's conjecture (Aizerman, 1949) states that the feedback loop is absolutely stable, this is, the origin is globally AS (GAS) for any non linearity $\psi \in [k_1, k_2]$.

Furthermore, Kalman's conjecture (Kalman, 1957) states, broadly speaking, that the system (3)-(5) is absolutely stable if the following conditions are fulfilled:

K.I. The function $\psi(y)$ is differentiable and such that $k_1 < \psi'(y) < k_2$.

K.II. All linear feedback loops with $\psi(y) = ky$, $k \in [k_1, k_2]$ are AS.

As it is well-known, both conjectures are false. However, their importance lies in the theoretical point of view, since they lead to the question 'What additional properties does the linear system have to satisfy in order for the conjectures to be true?' An answer to this question is provided by the circle criterion in the frequency domain, which relates to the KYP lemma in the state-space formulation.

3.1 Homogeneous extension of Lur'e problem

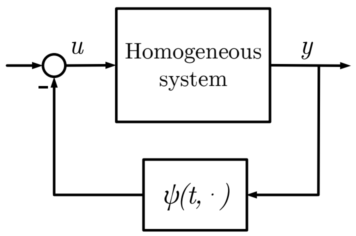


Figure 1. Lur'e-like system with a homogeneous block in the forward path.

Consider a generalisation of the Lur'e system (3)-(5):

$$\dot{x} = f(x) + g(x)u, \tag{8}$$
$$y = h(x) \tag{9}$$
$$u = -\psi(t, y) \tag{10}$$

as it is shown in Fig. 1, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ represents the states of the block in the forward path, whereas u and y, the input and output of the block, respectively, are scalar quantities.

The vector field on the right hand side of (8) is considered to be continuous in x and homogeneous of degree $\tau \in \mathbb{R}$ with respect to the dilations $\delta_\varepsilon^r x$ and $\varepsilon^s u$.

Notice that, since the homogeneous feedforward block is affine in the input, it is required that $f(x) \in \underline{n}_\tau$ and $g(x) \in \underline{n}_{\tau-s}$. Furthermore, $h(x)$ in (9) is a continuous homogeneous function of degree $\sigma > 0$ with respect to $\delta_\varepsilon^r x$ (i.e. $h \in H_\sigma$).

Finally, the feedback interconnection (10) is given by a memoryless function satisfying a *homogeneous sector condition.*

This new sector condition is proposed by following the logic of Aizerman's conjecture, in such a way that the feedback function ψ is chosen in order for the closed loop to preserve the homogeneity of degree τ of the block in the forward path. This is achieved by making $\psi(t, y) = \gamma\phi(y)$, $\gamma \in \mathbb{R}$, with $\phi(y)$ a homogeneous function of degree s/σ with respect to the dilation εy, which implies $u(\delta_\varepsilon^r x) = \varepsilon^s u(x)$. Therefore, from (2), the δ_ε^r-homogeneity of degree τ is preserved.

Definition 5. Let $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. It is said that ψ belongs to the *homogeneous sector*

- $[k_1, k_2]_\phi$, if $\psi(t, y)$ satisfies
$$\begin{aligned}(\psi(t, y) - k_1\phi(y))\,(\psi(t, y) - k_2\phi(y)) \le 0 \\ \forall\, t \in \mathbb{R}_+, \quad \forall\, y \in \mathbb{R},\end{aligned} \tag{11}$$
where k_1, k_2 are constants satisfying $k_1 < k_2$, and ϕ is a homogeneous function of degree s/σ, $\sigma > 0$, with respect to $\varepsilon > 0$.

- $[k_1, \infty]_\phi$, if $\psi(t, y)$ satisfies
$$\begin{aligned}\phi(y)\,(\psi(t, y) - k_1\phi(y)) \ge 0 \\ \forall\, t \in \mathbb{R}_+, \quad \forall\, y \in \mathbb{R}.\end{aligned} \tag{12}$$
where $k_1 \in \mathbb{R}$ and ϕ is a homogeneous function of degree s/σ, $\sigma > 0$, with respect to $\varepsilon > 0$.

The background presented thus far makes it possible to state the absolute stability problem for system (8)-(10) as follows:

Definition 6. Suppose that ψ in (10) satisfies a homogeneous sector condition. The closed loop system (8)-(10) is said to be *absolutely stable* if the origin is a GAS equilibrium.

Sufficient conditions for absolute stability of system (8)-(10) are presented in the following section.

4. HOMOGENEOUS CIRCLE CRITERION

Proposition 7. The closed-loop system (8)-(10) is absolutely stable if

(i) $\psi \in [k_1, \infty]_\phi$ and there exist a homogeneous Lyapunov function $V(x) \in H_m$, $m > \max_{1 \le i \le n} r_i$, real functions $M(x)$, and $L(x)$ such that $L^2(x)$ is positive definite, satisfying the following equations
$$\nabla V(x)\,(f(x) - k_1 g(x)\phi(h(x))) = -L^2(x) \tag{13}$$
$$\nabla V(x) g(x) = M^2(x)\phi(h(x)) \tag{14}$$

(ii) $\psi \in [k_1, k_2]_\phi$, with $k = k_2 - k_1$, and there exist a homogeneous Lyapunov function $V(x) \in H_m$, $m > \max_{1 \le i \le n} r_i$, $L : \mathbb{R}^n \to \mathbb{R}^\eta$, such that $L^T(x)L(x)$ is a positive definite function, and $M : \mathbb{R}^n \to \mathbb{R}^\eta$ for some value of η, which satisfy
$$\nabla V(x)\,(f(x) - k_1 g(x)\phi(h(x))) = -L^T(x)L(x) \tag{15}$$
$$\nabla V(x) g(x) = k\phi(h(x))M^T(x)M(x) - 2L^T(x)M(x) \tag{16}$$

Notice that the difference between this Proposition and the conditions for absolute stability derived from the dissipativity result by Hill and Moylan is the introduction of the homogeneous function or array of functions $M(x)$. In Section 6 this difference is further developed.

Proof.

(i) The derivative of V along the trajectories of system (8)-(10) is given by
$$\begin{aligned}\dot{V} &= \nabla V\,(f - g\psi(t, h)) \\ &= \nabla V(f - k_1 g\phi(h)) - \nabla V g\,(\psi(t, h) - k_1\phi(h)) \\ &= -L^2 - M^2\phi(h)\,(\psi(t, h) - k_1\phi(h))\end{aligned}$$

The fact that $\psi \in [k_1, \infty]_\phi$, i.e. $\phi(h)\,(\psi(t, h) - k_1\phi(h)) \ge 0$, implies $\dot{V} \le -L^2(x) < 0$. This allows us to conclude that the origin is a GAS equilibrium of the closed-loop system.

(ii) The derivative of V along the trajectories of system (8)-(10) is given by
$$\begin{aligned}\dot{V} &= \nabla V(f - g\psi(t, h)) \\ &= \nabla V(f - k_1 g\phi(h)) - \nabla V g\,(\psi(t, h) - k_1\phi(h)) \\ &= -L^T L - \left(k\phi(h)M^T M - 2L^T M\right)(\psi(t, h) - k_1\phi(h)) \\ &= -\|L - M(\psi(t, h) - k_1\phi(h))\|^2 \\ &\quad - M^T M\,(k_2\phi(h) - \psi(t, h))\,(\psi(t, h) - k_1\phi(h))\end{aligned}$$

where $\|\cdot\|$ is the Euclidean norm. Given that $\psi \in [k_1, k_2]_\phi$, $\dot{V} \le -\|L - M(\psi(t, h) - k_1\phi(h))\|^2 < 0$. This allows us to conclude that the origin is a GAS equilibrium of the closed-loop system. ■

5. EXAMPLES

Example 1. Consider the feedback loop

$$\Sigma : \begin{cases} \dot{x}_1 = -a\lfloor x_1\rceil^{\rho} + x_2, \\ \dot{x}_2 = -b\lfloor x_1\rceil^{2\rho-1} + u \\ y = h(x) = x_2 \end{cases} \quad (17)$$

$$u = -\psi(t,y), \quad \psi \in [k_1, \infty]_{\phi} \quad (18)$$

$$\phi(y) = \lfloor y\rceil^{\frac{2\rho-1}{\rho}} \quad (19)$$

where $\lfloor\cdot\rceil^q = |\cdot|^q\text{sign}(\cdot)$, $q \geq \mathbb{R}$. The vector field $\dot{x} = (\dot{x}_1, \dot{x}_2)^T = f(x)+g(x)u$ is homogeneous of degree $\tau = (\rho-1)/\rho$ with respect to the homogeneity weights $(r_1, r_2) = (1/\rho, 1)$ and $s = (2\rho-1)/\rho$. Moreover, the output $y = h(x)$ is a homogeneous function of degree $\sigma = 1$.

The case $\rho = 1/2$ corresponds to the *Super Twisting* algorithm (STA) (Levant, 1993), which has properties of robustness, precision and convergence to the origin in finite time. This algorithm is the basis for the construction of controllers, observers (Davila et al., 2005), and differentiators (Levant, 1998), in the frame of higher order sliding-mode control.

The sufficient conditions for stability introduced in the previous section can be used in order to find a region in the space of parameters a, b for which there exists k_1 such that the closed-loop system (17)-(19) is absolutely stable. Moreover, the values of k_1 can also be found following this strategy.

Theorem 1 suggests a homogeneous Lyapunov candidate function. In order for the function to be of class C^1, we make m equal to 2. Given the homogeneity weights, the following function appears as a natural candidate:

$$V = \frac{b}{2\rho}|x_1|^{2\rho} + \frac{1}{2}|x_2|^2 \quad (20)$$

This function is positive definite if the constant b is positive.

Let us find a system of inequalities that guarantees the conditions in (13) and (14) are met. On the one hand, condition (16) is satisfied if

$$\nabla V(x)g(x) = x_2 = \lfloor x_2\rceil^{\frac{2\rho-1}{\rho}} M^2(x).$$

This is achieved by making $M^2(x) = |x_2|^{\frac{1-\rho}{\rho}}$. On the other hand, we find that condition (13) is fulfilled if $-\nabla V(f - k_1 g\phi(h)) = L^2$ is a positive definite function, this is

$$\begin{aligned} L^2 &= (a\lfloor x_1\rceil^{\rho} - x_2)\, b\lfloor x_1\rceil^{2\rho-1} \\ &\quad + \left(b\lfloor x_1\rceil^{2\rho-1} + k_1\lfloor x_2\rceil^{\frac{2\rho-1}{\rho}}\right) x_2 \quad (21) \\ &= ab|x_1|^{3\rho-1} + k_1|x_2|^{\frac{3\rho-1}{\rho}}, \end{aligned}$$

which is, clearly, positive definite if $a > 0$ and $k_1 > 0$. Therefore, it is possible to assert the absolute stability of system (17)-(19) for such values of a, b, and k_1.

It should be noted that the Lyapunov function (20) is of class C^1 if $\rho > 1/2$. In order to conclude on the absolute stability of the STA ($\rho = 1/2$), tools of nonsmooth analysis can be used as those in Shevitz and Paden (1994), Clarke (1990), and Bacciotti and Ceragioli (1999).

Example 2. Let us consider now the same system with a different output, as follows

$$\Sigma : \begin{cases} \dot{x}_1 = -a\lfloor x_1\rceil^{\rho} + x_2, \\ \dot{x}_2 = -b\lfloor x_1\rceil^{2\rho-1} + u \\ y = h(x) = x_1 \end{cases} \quad (22)$$

$$u = -\psi(t,y), \quad \psi \in [k_1, k_2]_{\phi} \quad (23)$$

$$\phi(y) = \lfloor y\rceil^{2\rho-1} \quad (24)$$

The vector field is again homogeneous of degree $\tau = (\rho-1)/\rho$ with respect to the homogeneity weights $(r_1, r_2) = (1/\rho, 1)$ and $s = (2\rho-1)/\rho$. However, the output $y = h(x) = x_1$ is a homogeneous function of degree $\sigma = 1/\rho$.

Conditions (15) and (16) can be used in order to find a region in the space of parameters a, b for which there exist k_1 and k_2 such that the feedback loop (22)-(24) is absolutely stable. Following a similar approach to that of the previous example, the following homogeneous Lyapunov candidate function (Moreno et al., 2014) is proposed:

$$V = \frac{\gamma_1}{3\rho}|x_1|^{3\rho} - \lfloor x_1\rceil^{2\rho}x_2 + \frac{\gamma_2}{3}|x_2|^3.$$

In order for this function to be positive definite, the inequalities

$$\gamma_1 - 2\rho c^{3/2} > 0 \quad \text{and} \quad \gamma_2 - c^{-3} > 0 \quad (25)$$

have to be satisfied, for some $c > 0$. This is due to the fact that for any real numbers $\alpha > 0$, $\beta > 0$, $c > 0$, $p > 1$, and $q > 1$ with $p^{-1} + q^{-1} = 1$, the following inequality (Young's inequality) holds

$$\alpha\beta < \frac{c^p}{p}\alpha^p + \frac{c^{-q}}{q}\beta^q.$$

Therefore,

$$V \geq \frac{1}{3\rho}\left(\gamma_1 - 2\rho c^{3/2}\right)|x_1|^{3\rho} + \frac{1}{3}\left(\gamma_2 - c^{-3}\right)|x_2|^3.$$

The inequalities in (25) are equivalent to $\gamma_1 > 2\rho c^{3/2}$ and $c^3 > \frac{1}{\gamma_2}$. Thus, a necessary and sufficient condition for the positivity of V is that $\gamma_2\gamma_1^2 > 4\rho^2$.

Let us find a system of inequalities that guarantees the conditions in (15) and (16) are met. On the one hand, condition (16) is satisfied if

$$\begin{aligned} \nabla V(x)g(x) &= -\lfloor x_1\rceil^{2\rho} + \gamma_2\lfloor x_2\rceil^2 \\ &= \left(k\lfloor x_1\rceil^{2\rho-1}M(x) - 2L(x)\right)^T M(x). \end{aligned}$$

This can be achieved by making

$$M(x) = k^{-1/2}\begin{pmatrix} |x_1|^{1/2} \\ \gamma_2^{1/2}|x_1|^{\frac{1-2\rho}{2}}|x_2| \\ 0 \end{pmatrix}, \quad \text{and}$$

$$L(x) = \begin{pmatrix} k^{1/2}\lfloor x_1\rceil^{\frac{4\rho-1}{2}} \\ \dfrac{\gamma_2^{1/2}k^{1/2}}{2}\left(\lfloor x_1\rceil^{\frac{2\rho-1}{2}}|x_2| + |x_1|^{\frac{2\rho-1}{2}}x_2\right) \\ L_3(x) \end{pmatrix}.$$

On the other hand, we find that condition (15) is fulfilled if $L_3^2 = -\nabla V(f - k_1 g\phi(h)) - L_1^2 - L_2^2$ is a positive definite function, this is

$$\begin{aligned} L_3^2 &= (a\lfloor x_1\rceil^{\rho} - x_2)\left(\gamma_1\lfloor x_1\rceil^{3\rho-1} - 2\rho|x_1|^{2\rho-1}x_2\right) \\ &\quad + (b + k_1)\lfloor x_1\rceil^{2\rho-1}\left(\gamma_2\lfloor x_2\rceil^{2} - \lfloor x_1\rceil^{2\rho}\right) \\ &\quad - \gamma_{12}k|x_1|^{4\rho-1} - \frac{k\gamma_2}{2}|x_1|^{2\rho-1}|x_2|^2 \\ &\quad + \frac{k\gamma_2}{2}\lfloor x_1\rceil^{2\rho-1}\lfloor x_2\rceil^{2} \\ &= |x_1|^{2\rho-1}\begin{pmatrix}\lfloor x_1\rceil^{\rho} \\ x_2\end{pmatrix}^T Q(x)\begin{pmatrix}\lfloor x_1\rceil^{\rho} \\ x_2\end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} Q(x) &= \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22}(x)\end{pmatrix} \\ &= \begin{pmatrix} a\gamma_1 - (b + k_1 + k) & -\dfrac{\gamma_1 + 2\rho a}{2} \\ -\dfrac{\gamma_1 + 2\rho a}{2} & Q_{22}(x)\end{pmatrix}, \end{aligned}$$

with

$$Q_{22}(x) = 2\rho - \gamma_2\left(\frac{k}{2} - \left(b + k_1 + \frac{k}{2}\right)\lfloor x_1 x_2\rceil^{0}\right).$$

Observe that for $x_1x_2 < 0$, $L_3^2(x)$ is positive if and only if $Q_{11} > 0$ and $2\rho - \gamma_2(b + k_1 + k) > 0$.

Notice also that $L_3^2(x)$ is positive if and only if $Q(x)$ is a positive definite matrix for $x_1x_2 > 0$ or, equivalently, $Q_{11} > 0$ and

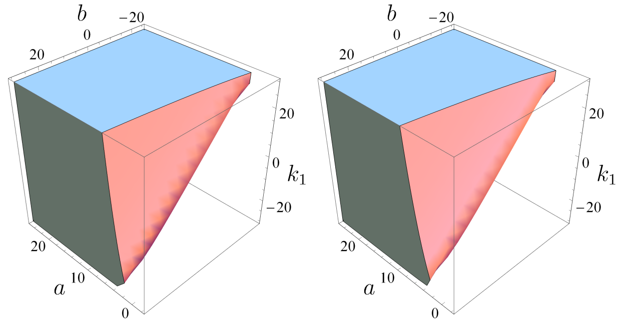
$$Q_{11}(2\rho + \gamma_2(b + k_1)) > \left(\frac{\gamma_1 + 2\rho a}{2}\right)^2.$$

In summary, system (17)-(19) is absolutely stable if there exist $k_1 \in \mathbb{R}$, $k \geq 0$, γ_1, and γ_2 such that following inequalities are satisfied:

$$\begin{aligned} \gamma_2\gamma_1^2 &> 4\rho^2 \\ Q_{11} = a\gamma_1 - (b + k_1 + k) &> 0 \\ Q_{11}(2\rho + \gamma_2(b + k_1)) &> \frac{1}{4}(\gamma_1 + 2\rho a)^2 \\ 2\rho - \gamma_2(b + k_1 + k) &> 0 \end{aligned} \quad (26)$$

Notice that when the inequalities in (26) are true, $L^2(x)$ is positive definite only if $\rho = 1/2$. For other values of ρ, it is positive semidefinite. However, for the case in which ψ does not depend directly on t, asymptotic stability still holds due to the invariance principle.

A plot of the region of parameters that renders system (17)-(19) absolutely stable can be found using inequalities (26), as shown in Fig. 2.



(a) Case $\rho = 1$ (b) Case $\rho = 1/2$

Figure 2. Regions of absolute stability in the space of parameters for different values of ρ and $k = 2$.

Therefore, the circle criterion brings a two-way connection between the solution to the Lur'e problem and the concept of passivity, which is a recurring theme in the literature of non linear systems theory. However, the generalisation that we propose does not have such an immediate connection to conditions for passivity as those of Hill-Moylan; the difference comes from the introduction of the term $M(x)$.

Making $M(x) = 1$, that is if conditions (15), (16) were instead

$$\nabla V(x)\left(f(x) - k_1 g(x)\phi(h(x))\right) = -L^2(x) \quad (27)$$

$$\nabla V(x)g(x) = k\phi(h(x)) - 2L(x), \quad (28)$$

then, the conditions are equivalent to the cited Hill-Moylan conditions of passivity for the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)(\bar{u} - k_1\phi(h(x))) \\ \bar{y} &= k\phi(h(x)) + \bar{u}, \end{aligned}$$

which is a loop transformation as those found in the literature of nonlinear systems regarding the absolute stability problem (Vidyasagar, 2002) (Khalil, 1996). However, if these conditions were satisfied there would be an undesirable restriction on the degree of homogeneity of $V(x)$.

To illustrate this, let us take the case of Example 2. The conditions in (27) would force the homogeneity degree of $V(x)$ (which we denote as m) to be equal to $2s - \tau$. This is because equation (28) and the fact that V, g, $\phi(h)$, and L are δ_ε^r homogeneous of degrees m, $\tau - s$, s, and s, respectively, imply that the homogeneity degree of $\nabla V(x)g(x)$ is equal to that of $\phi(h(x))$. This is $m + \tau - s = s$. In this case $s = (2\rho - 1)/\rho$, and $\tau = (\rho - 1)/\rho$. Therefore, the degree of homogeneity of V would necessarily be equal to $2s - \tau = (3\rho - 1)/\rho$.

As the interest of this article is to work with functions as those described by Theorem 1, the recently found value of m allows to propose functions of class C^1 only if $\rho > 2/3$ (because this requires m to be greater than $\max\{1/\rho, 1\}$, therefore $(3\rho - 1)/\rho > 1/\rho$ which implies $\rho > 2/3$). Introducing $M(x)$ allows to choose an arbitrary homogeneity degree for the function $V(x)$, which translates to V being differentiable an arbitrary number of times.

Equally remarkable is the fact that the homogeneous sector condition introduced in Section 3 enables the analysis of closed-loop systems with feedback functions having different properties, for instance, non-Lipschitz functions at

6. DISCUSSION

As it was stated in Section 3, the circle criterion that provides a solution to the Lur'e problem has a close relationship with the prominent KYP lemma. This lemma, in turn, offers a tool in the state-space to find out whether or not a system is (strictly) positive real, a property intimately related to the concept of passivity.

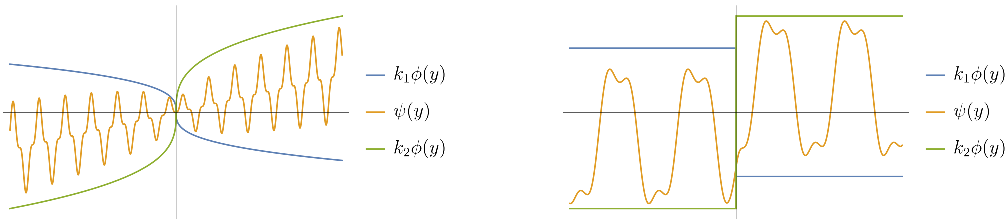


Figure 3. Feedback functions ψ in different sectors.

zero or functions that do not cross the origin, as shown in Fig. 3.

It is also important to emphasise the relevance of this result concerning the analysis of complex non-linear and non-homogeneous systems exhibiting properties such as finite-time stabilisation, as long as they can be represented by the generalised Lur'e system that is proposed. Moreover, this is also a tool for the study of the stability of time-varying homogeneous systems, see, for instance, Jerbi et al. (2013) and Peuteman and Aeyels (1999).

7. CONCLUSIONS

The aim of this paper is to develop a set of tools in order to analyse the stability of a class of systems that can be modelled by the feedback loop (8)-(10), in similar terms as those that offer solutions to the Lur'e problem.

The main contributions presented are the statement of homogeneous generalisations of the Lur'e problem and the circle criterion.

Contrary to the original circle criterion, the conditions presented in this article do not have an immediate connection to the results of passivity and dissipativity, which motivates further research.

REFERENCES

Aizerman, M.A. (1949). On a problem concerning the stability "in the large" of dynamical systems. *Uspekhi matematicheskikh nauk*, 4(4), 187–188.

Bacciotti, A. and Ceragioli, F. (1999). Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions. *ESAIM: Control, Optimisation and Calculus of Variations*, 4, 361–376.

Bacciotti, A. and Rosier, L. (2006). *Liapunov functions and stability in control theory*. Springer Science & Business Media.

Bhat, S.P. and Bernstein, D.S. (1997). Finite-time stability of homogeneous systems. In *American Control Conference, 1997. Proceedings of the 1997*, volume 4, 2513–2514. IEEE.

Brogliato, B. and Heemels, W.M.H. (2009). Observer design for Lur'e systems with multivalued mappings: A passivity approach. *IEEE Transactions on Automatic Control*, 54(8), 1996–2001.

Clarke, F.H. (1990). *Optimization and nonsmooth analysis*. SIAM.

Davila, J., Fridman, L., and Levant, A. (2005). Second-order sliding-mode observer for mechanical systems. *IEEE Transactions on Automatic Control*, 50(11), 1785–1789.

Grüne, L. (2000). Homogeneous state feedback stabilization of homogenous systems. *SIAM Journal on Control and Optimization*, 38(4), 1288–1308.

Hermes, H. (1991). Nilpotent and high-order approximations of vector field systems. *SIAM review*, 33(2), 238–264.

Hill, D. and Moylan, P. (1976). The stability of nonlinear dissipative systems. *IEEE Transactions on Automatic Control*, 21(5), 708–711.

Jerbi, H., Kharrat, T., and Kallel, W. (2013). Some results on stability and stabilization of homogeneous time-varying systems. *Mediterranean Journal of Mathematics*, 1–12.

Kalman, R.E. (1957). Physical and mathematical mechanisms of instability in nonlinear automatic control systems. *Trans. ASME*, 79(3), 553–566.

Kawski, M. (2015). Homogeneity in control: Geometry and applications. In *European Control Conference (ECC), 2015*, 2449–2457. IEEE.

Khalil, H.K. (1996). Noninear systems. *Prentice-Hall, New Jersey*, 2(5).

Levant, A. (1993). Sliding order and sliding accuracy in sliding mode control. *International Journal of Control*, 58(6), 1247–1263.

Levant, A. (1998). Robust exact differentiation via sliding mode technique. *Automatica*, 34(3), 379–384.

Levant, A. (2005). Homogeneity approach to high-order sliding mode design. *Automatica*, 41(5), 823–830.

Lur'e, AI y Postnikov, V. (1944). On the theory of stability of control systems. *Applied mathematics and mechanics*, 8(3), 246–248.

Moreno, J., Sánchez, T., and Cruz-Zavala, E. (2014). Una función de Lyapunov suave para el algoritmo super-twisting. *16th CLCA*, 182–187.

Peuteman, J. and Aeyels, D. (1999). Averaging results and the study of uniform asymptotic stability of homogeneous differential equations that are not fast time-varying. *SIAM journal on control and optimization*, 37(4), 997–1010.

Rosier, L. (1992). Homogeneous Lyapunov function for homogeneous continuous vector field. *Systems & Control Letters*, 19(6), 467–473.

Sepulchre, R. and Aeyels, D. (1996). Homogeneous Lyapunov functions and necessary conditions for stabilization. *Mathematics of Control, Signals and Systems*, 9(1), 34–58.

Shevitz, D. and Paden, B. (1994). Lyapunov stability theory of nonsmooth systems. *IEEE Transactions on automatic control*, 39(9), 1910–1914.

Vidyasagar, M. (2002). *Nonlinear systems analysis*. SIAM.