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Interconnection and damping assignment for implicit port-Hamiltonian systems ★

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Abstract: Implicit port-Hamiltonian representations of mechanical systems are considered from a control perspective. Energy shaping is used for the purpose of stabilizing a desired equilibrium. When using implicit models, the problem turns out to be a simple quadratic programming problem (as opposed to the partial differential equations that need to be solved when using explicit representations).

Keywords: Hamiltonian Dynamics, Holonomic Constraints, Implicit Models, Passivity, Pendulum, Cart-Pole

1. INTRODUCTION

The Hamiltonian formalism is used to describe the dynamics of a wide class of systems including mechanical (Arnold et al., 2006; van der Schaft and Maschke, 1994), electrical (Maschke et al., 1995; Bernstein and Liberman, 1989; Blankenstein, 2005; Castaños et al., 2009), and thermodynamic (Öttinger, 2005; Sandberg et al., 2011) ones.

In many cases there are constraints imposed on the system coordinates. These constraints reflect the internal structure of the system, for instance, rigid connections between the system's elements. From the geometrical viewpoint, the action of these constraints results in restricting the system's evolution to a submanifold of the state space.

When the system is subject to the action of external forces it is convenient to consider a pair of (energy-adjoint) port variables (u, y) such that their product is equal to the power supplied into the system. Such model is referred to as a port-Hamiltonian system (see Maschke and van der Schaft (1992) for the original definition as stated with respect to Hamiltonian systems in explicit form).

In general, there are two different approaches to the representation of systems evolving on manifolds: the explicit representation with the dynamics having the form of an ordinary differential equation on the manifold and the implicit representation with the dynamics described by a set of differential-algebraic equations usually evolving in a Euclidean space (see, e.g., Castaños et al. (2013) for a related discussion on constrained Hamiltonian systems). There has been a lot of research on the analysis and control of explicit systems (van der Schaft, 2000; Ortega et al., 2001). However, not many results on the control of Hamiltonian systems in implicit formulation have been presented so far. Thus, the primary goal of this contribution is to provide an elaborated approach to the control of Hamiltonian systems in implicit form.

We note that there is a series of papers presenting a unified approach to the description and analysis of implicit Hamiltonian systems on the base of (generalized) Dirac structures, e.g., (van der Schaft, 1998; Dalsmo and van der Schaft, 1999). It has been shown that Dirac structures can be used for the analysis of symmetries (Blankenstein and van der Schaft, 2001), and interconnection properties (Cervera et al., 2007) of (implicit) Hamiltonian systems (see also the book (Duindam et al., 2009) for more details). Recently, there has been a paper devoted to the control of (discretized) infinite-dimensional implicit Hamiltonian systems, (Macchelli, 2014). However, the authors feel that while Dirac structures offer a unified approach it is sometimes more advantageous to have a closer look at the object under study. In this sense, the approach presented in this paper allows one to consider the problem at hand at a practical level, without a (sometimes) unnecessary generalization.

The paper is organized as follows: in Section 2, an implicit representation of port-Hamiltonian systems is presented and a couple of simple models are derived within the described framework. In Section 3, the energy shaping approach is presented in details and a number of illustrative examples is given. Finally, Section 4 presents the conclusions and the directions for future research.

2. IMPLICIT PORT-HAMILTONIAN SYSTEMS

2.1 Mechanical systems with holonomic constraints

Consider a controlled mechanical system with the Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$. Let there be a number of *holonomic* constraints $c(r) = 0$, $c : \mathbb{R}^n \to \mathbb{R}^k$, restricting the configuration space of the system to an $(n-k)$-dimensional submanifold Γ of the configuration space $\mathbb{R}^n$. Using the Hamiltonian formalism, the dynamics of this system is

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described by a set of differential-algebraic equations of the form (Hairer et al., 2006; Castaños et al., 2013):

$$\dot{x} = J\left(\nabla H(x) + \nabla c(x)\lambda\right) + \hat{g}(x)u \tag{1a}$$

$$0 = c(x) \tag{1b}$$

$$y = \nabla H^\top(x)\hat{g}(x) \ , \tag{1c}$$

where the state is given by $x^\top = \begin{pmatrix} r^\top & p^\top \end{pmatrix}$ with $r \in \mathbb{R}^n$ and $p \in \mathbb{R}^{*n}$ the positions and momenta, respectively,

$$\nabla c(x) = \frac{\partial c^\top}{\partial x}(x)$$

is the transposed Jacobian of the vector-valued function $c(x)$, $\lambda \in \mathbb{R}^k$ is the vector of implicit variables that enforce the holonomic constraints, $(u, y) \in \mathbb{R}^{*m} \times \mathbb{R}^m$ are the conjugated external port variables, and $\hat{g}(x) = \begin{pmatrix} 0_{[m\times n]} & g^\top(x) \end{pmatrix}^\top$ is a $(2n \times m)$-matrix such that $\operatorname{rank} \hat{g}(x) = m$ for all $x \in \mathbb{R}^{2n}$. The $[2n \times 2n]$-matrix J is the one associated to the canonical symplectic form,

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \ .$$

Here and forth all functions are assumed to be smooth enough and the gradient is assumed to be a column vector.

Equations (1) correspond to a port-Hamiltonian system (van der Schaft, 2000; Dalsmo and van der Schaft, 1999) with an augmented Hamiltonian function $\tilde{H}(x) = H(x) + c(x)\lambda$ (see Arnold et al. (2006, p. 48) for a more general treatment).

From the geometrical viewpoint, (1) describe the system evolution on the cotangent bundle of $\mathbb{R}^n$, denoted $T^*\mathbb{R}^n$. The vector field $X \in T\left(T^*\mathbb{R}^n\right)$ can be written as

$$X = D_H + D_\lambda \lambda + D_g u \tag{2a}$$

$$0 = c \tag{2b}$$

$$y = D_g(H) \ , \tag{2c}$$

where, with Einstein's summation convention implied,

$$D_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial r^i} - \frac{\partial H}{\partial r^i}\frac{\partial}{\partial p_i}$$

is the Hamiltonian vector field,

$$D_\lambda \lambda = -\frac{\partial c^j}{\partial r^i}\lambda_j \frac{\partial}{\partial p_i}$$

is the vector field of the internal (constraint) forces, and

$$D_g u = g_i^j u_j \frac{\partial}{\partial p_i}$$

is the control vector field. Note that D_λ and D_g are the tuples of linearly independent vector fields: $D_\lambda = \left(D_\lambda{}^1, \ldots, D_\lambda{}^k\right)$, and $D_g = \left(D_g{}^1, \ldots, D_g{}^m\right)$. Thus, for instance, the application of D_λ to a smooth function $f(x)$ yields a vector $D_\lambda f(x) = \left(D_\lambda{}^1 f(x), \ldots, D_\lambda{}^k f(x)\right)$.

Equation (1b) constrains the configuration space of (1). We wish to assure that these constraints are preserved under the system dynamics. To do so we require X to be tangential to Γ, i.e., $X(c^i) = 0$ for any $i = 1, \ldots, k$. This yields the so-called hidden constraints,

$$G^i(x) = X(c^i) = \frac{\partial H(x)}{\partial p_j}\frac{\partial c^i(x)}{\partial r^j} = 0 \ . \tag{3}$$

Now, considering $T^*\mathbb{R}^n$ as a state space manifold, we say that (1) evolves on a submanifold $\mathcal{M}_\Gamma \subset T^*\mathbb{R}^n$,

$$\mathcal{M}_\Gamma = \{x = (r,p) \in \mathbb{R}^n \times \mathbb{R}^{*n} | c^i(x) = 0, \\ G^i(x) = 0, \ i = 1, \ldots, k\} \ .$$

Assumption 1. The following holds:

i) The constraints are regular, i.e.,
$$\dim \operatorname{span}\left\{dc^i(r)\right\}_{r\in\Gamma} = k \ ,$$
where $dc^i(r) \in T^*\mathbb{R}^n$ are the differentials of $c^i(r)$ interpreted as the elements of the cotangent vector bundle $T^*\mathbb{R}^n$. Note that the manifold Γ is an integral manifold of the distribution generated by dc_i, i.e.,
$$T\Gamma = \ker\left(\operatorname{span}\left\{dc^i\right\}\right) \ , \quad i = 1, \ldots, k$$

ii) The initial conditions belong to $\mathcal{M}_\Gamma$, i.e.,
$$x(0) = (r(0), p(0)) \in \mathcal{M}_\Gamma \ .$$

iii) The energy is separable and positive definite w.r.t. p, i.e.,
$$H(x) = P(r) + K(p) \ , \ K(p) = \frac{1}{2}p^\top M^{-1} p \ , \ M > 0 \ ,$$
where P and K are the potential and kinetic energy, respectively.

Assumptions i) and iii) guarantee that $\mathcal{M}_\Gamma$ is a proper subbundle of $T^*\mathbb{R}^n$. Indeed, for any $r \in \Gamma$, the hidden constraints define a linear subspace of codimension k, which is interpreted as the cotangent subspace to Γ at x.

Item i) and strict convexity in iii) ensure that the λ_i exist and are uniquely defined. More precisely, applying the vector field to the hidden constraints yields the condition

$$X(G) = D_H^2(c) + D_\lambda D_H(c)\lambda + D_g D_H(c) u = 0 \ , \tag{4}$$

which implicitly defines λ as a function of x and u. Notice that the $(k \times k)$-matrix defined by

$$D_\lambda D_H(c) = D_\lambda(G) = D_\lambda\left(\frac{\partial H}{\partial p_i}\frac{\partial c}{\partial r^i}\right) = -\frac{\partial^2 H}{\partial p_i \partial p_j}\frac{\partial c^a}{\partial r^i}\frac{\partial c^b}{\partial r^j} \tag{5}$$

is negative definite as follows from Assumptions i) and iii) and hence, invertible. This ensures the well-posedness of the problem.

Assumption ii) guarantees that there are no jumps in the system's trajectories.

Finally, separability of the Hamiltonian in item iii) is, from a computational point of view, one of the main advantages of the implicit modeling framework (see, e.g., Castaños et al. (2015)).

Note that the hidden constraints (3) imply that the Hamiltonian is invariant under the action of the vector field of constraint forces, i.e.,

$$D_\lambda(H) = 0 \ .$$

This is equivalent to saying that the internal forces do not produce work and hence do not alter the total energy of the system. Furthermore, the vector field D_λ is also tangential to the submanifold Γ, i.e.,

$$D_\lambda(c) = 0 \ . \tag{6}$$

To get more insight into the nature of the vector field of internal forces we recall that the cotangent bundle $T^*\mathbb{R}^n$ is endowed with the canonical symplectic form $\omega = dr^i \wedge dp_i$. The symplectic form defines a canonical isomorphism between the tangent and cotangent spaces: $\Omega : X \mapsto \omega(X, \cdot)$. It can be easily seen that the vector fields $D_{\lambda i}$ are isomorphic to the covector fields dc^i.

Now we can establish a relationship between two seemingly unrelated conditions: fulfillment of hidden constraints, $X(c^i) = 0$, and the invariance of the Hamiltonian H under D_λ, $D_\lambda(H) = 0$. First we note that, since c^i depend only on r, we can write $X(c^i) = D_H(c^i)$. Then we have the following:

$$D_H(c^i) = dc^i(D_H) = \omega(D_H, \Omega^{-1}(dc^i)) = \\ \omega(D_H, {D_\lambda}^i) = dH({D_\lambda}^i) = {D_\lambda}^i(H) = 0 \ .$$

Now we will show that the passivity property which is central for port-Hamiltonian systems can be readily extended to the case of the constrained dynamics (1).

Proposition 2. Consider the *restricted state-space* $\mathcal{M}_\Gamma$. System (1) is passive whenever $H|_{\mathcal{M}_\Gamma}$, the restriction of H to $\mathcal{M}_\Gamma$, is bounded from below.

Proof: Taking the derivative of H gives

$$\dot{H} = X(H) = D_g(H)u = g_i^j \frac{\partial H}{\partial p_i} u_j = y^j u_j \ .$$

This equation, together with the lower bound on H, implies passivity. □

Finally, we give a condition for a constrained system to be fully actuated.

Definition 3. We say that (1) is fully actuated whenever

$$\operatorname{span}\left\{ \begin{pmatrix} 0_n \\ I_n \end{pmatrix} \right\} = \operatorname{span}\{D_\lambda\} \oplus \operatorname{span}\{D_g\}$$

for all $x \in \mathcal{M}_\Gamma$. We say that (1) is underactuated if it is not fully actuated.

2.2 A simple actuated pendulum

Consider a simple pendulum with mass m_1 held by an ideal massless bar of length l. Let $r^\top = (r^x \ r^y)$ and $p^\top = (p_x \ p_y)$ be the position and momenta, respectively. The constraint is given by $c^1(x) = \frac{1}{2}\left(\|r\|^2 - l^2\right) = 0$, while the energy takes the form

$$H(x) = \frac{1}{2m_1}\|p\|^2 + m_1\bar{g} \cdot r^y$$

with $\bar{g}$ the acceleration due to gravity. Suppose that a torque u_1 is applied to the pendulum axis. The implicit model then takes the form

$$\dot{r} = \frac{1}{m_1} p \tag{7a}$$

$$\dot{p} = -m_1\bar{g}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} r^x \\ r^y \end{pmatrix}\lambda_1 + \frac{1}{l^2}\begin{pmatrix} -r^y \\ r^x \end{pmatrix} u_1 \tag{7b}$$

$$y^1 = \frac{r_x p^y - r^y p_x}{m_1 l^2} \tag{7c}$$

It is not difficult to verify that Assumption 1 holds, and that the system is fully actuated.

Boundedness of H can be easily established. Given the positive definite form of K, it is only necessary to verify the term $m_1\bar{g}\cdot r^y$. The term is continuous and restricted to the compact set $\Gamma = \left\{ r \in \mathbb{R}^2 \mid \|r\| = l \right\}$. By the extreme value theorem of Weierstrass, we know that the term is bounded from below and the passivity of the pendulum is confirmed.

2.3 A pendulum on a cart

Consider now an actuated cart with mass m_1, position $r^1 \in \mathbb{R}^2$ and momentum $p_1 \in \mathbb{R}^2$. The cart is constrained to move along the x-axis, which can be expressed as $c^1(x) = 0$ with $c^1(x) = r^{1_y}$. Attached to the cart is a pendulum of length l, mass m_2, position $r_2 \in \mathbb{R}^2$ and momentum $p_2 \in \mathbb{R}^2$. The bond between the cart and the pendulum is expressed as $c^2(x) = 0$ with

$$c^2(x) = \frac{1}{2}\left(\|r^2 - r^1\|^2 - l^2\right) \ .$$

The total energy is given by

$$H(x) = \frac{1}{2m_1}\|p_1\|^2 + \frac{1}{2m_2}\|p_2\|^2 + m_2\bar{g}\cdot r^{2_y} \ , \tag{8}$$

so the pendulum takes the form

$$\begin{aligned} \dot{r}^1 &= \frac{1}{m_1}p_1 \\ \dot{r}^2 &= \frac{1}{m_2}p_2 \\ \dot{p} &= -m_2\bar{g}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & r^{1_x} - r^{2_x} \\ 1 & r^{1_y} - r^{2_y} \\ 0 & r^{2_x} - r^{1_x} \\ 0 & r^{2_y} - r^{1_y} \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u \\ y^1 &= \frac{p_{1_x}}{m_1}. \end{aligned} \tag{9}$$

Again, Assumption 1 holds, but the system is underactuated:

$$o + m = 2 + 1 < 4 = n \ .$$

The constraint $\|r^2 - r^1\| = l$ implies that $\|r^{2_y} - r^{1_y}\| \leq l$. Since $r_{1_y} = 0$, we have $\|r^{2_y}\| \leq l$, which defines a compact set on r^{2_y}. Weierstrass Theorem then implies that the restriction of $m_2\bar{g}\cdot r^{2_y}$ is bounded from below and the pendulum on a cart is passive as well.

Note that the described framework is general enough to model most classes of mechanical systems, including manipulators and various types of robotic arms such as the acrobot, the pendubot and many more.

3. IMPLICIT ENERGY SHAPING

3.1 The matching equations

Definition 4. Let H_d be a smooth mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}$. We say that H_d is an admissible energy (Hamiltonian) function if the matching equation

$$D_{H_\mathrm{d}} - D_H = D_\lambda\mu + D_g\hat{u} \tag{10}$$

with $D_{H_\mathrm{d}} = \Omega^{-1}(dH_\mathrm{d})$ is satisfied for some $\mu \in \mathbb{R}^k$, $\hat{u} \in \mathbb{R}^m$ and if

$$D_\lambda(H_d) = 0 \ . \tag{11}$$

Setting $u = \hat{u} + \upsilon$ and substituting (10) into (1) gives the new port-Hamiltonian system

$$\dot{x} = J\left(\nabla H_\mathrm{d}(x) + \nabla c(x)(\lambda - \mu)\right) + \hat{g}(x)\upsilon \tag{12a}$$

$$0 = c(x) \tag{12b}$$

$$y_\mathrm{d} = \nabla H_\mathrm{d}^\top(x)\hat{g}(x) \tag{12c}$$

with port variables $(\upsilon_j, y_{\mathrm{d}}^j)$.

Since $\lambda - \mu$ is an implicit variable, i.e. it is found as the solution to the auxiliary condition (4), the way it is denoted is immaterial and thus it is possible to rename $(\lambda - \mu)$ to λ without changing the system dynamics. This leads us to the following definition.

Definition 5. Given the vector fields of internal forces $\{D_\lambda{}^i\}$, $i = 1, \ldots, k$, two Hamiltonian functions H_1 and H_2, $H_i : \mathbb{R}^{2n} \to \mathbb{R}$, $i = 1, 2$, are said to be equivalent, $H_1(x) \sim_\lambda H_2(x)$ if

$$D_{H_1} - D_{H_2} \in span\left\{D_\lambda{}^i\right\},$$

where $D_{H_i} = \Omega^{-1}(dH_i)$. The equivalence class of H, denoted $[H]_\lambda$, is defined as $[H]_\lambda = \{\hat{H} : \mathbb{R}^{2n} \to \mathbb{R} \| \hat{H} \sim_\lambda H\}$.

We have the following proposition.

Proposition 6. Let $\hat{H}$ satisfy (10) and (11). Then, any $H_d \in [\hat{H}]_\lambda$ is an admissible energy function.

Proof: We need to prove that for any $H_{\mathrm{d}} \in [\hat{H}]_\lambda$ conditions (10) and (11) hold. An H_{d} can be represented as $H_{\mathrm{d}} = \hat{H} + \kappa_i c^i$, $i = 1, \ldots, k$. If $\hat{H}$ satisfies (10), then H_{d} satisfies (10) as well with $\hat{\mu} = \mu - \kappa$. Furthermore, (11) is satisfied as $D_\lambda(c) = 0$. □

This gives additional freedom for choosing H_{d} in (10). Roughly speaking, this additional freedom 'compensates' for the need to solve (10) in a high-dimensional setting (i.e., higher than in the explicit formulation) using the same number of controls $\hat{u}$.

Equation (11) is analogous to the one formulated for the original system. It ensures that the new Hamiltonian vector field $D_{H_{\mathrm{d}}}$ preserves the holonomic constraints c^i, i.e., $D_{H_{\mathrm{d}}}(c^i) = 0$ and that the constraint forces preserve the new energy, i.e., $D_\lambda^i(H_{\mathrm{d}}) = 0$ whenever $x \in \mathcal{M}_\Gamma$. See Sec. 2.1 for more details.

Proposition 7. If $H_{\mathrm{d}}|_{\mathcal{M}_\Gamma}$ is bounded from below, then the closed-loop (12) is passive and the storage function is equal to H_{d}.

Proof: Direct computation gives

$$\dot{H}_{\mathrm{d}} = D_g(H_{\mathrm{d}})u = g_i^j \frac{\partial H_{\mathrm{d}}}{\partial p_i} u_j = y_{\mathrm{d}}^j \upsilon_j \;.$$

□

3.2 Equilibrium stabilization

Let

$$x^* = \begin{pmatrix} r^* \\ 0 \end{pmatrix} \in \mathcal{M}_\Gamma \tag{13}$$

be a desired equilibrium point. It follows from standard Lyapunov theory that x^* is stabilizable whenever H_{d} is admissible and x^* is a strict minimum of $H_{\mathrm{d}}|_{\mathcal{M}_\Gamma}$,

$$\underset{x \in \mathcal{M}_\Gamma}{\arg\min}\, H_{\mathrm{d}}(x) = x^* \;. \tag{14}$$

The problem is easily solvable in the fully actuated case.

Theorem 8. Let (1) be fully actuated. Any x^* satisfying (13) is an assignable equilibrium and can be stabilized.

Proof: Set

$$H_{\mathrm{d}}(x) = a^\top r + \frac{1}{2}(r - r^*)^\top A (r - r^*) + \frac{1}{2} p^\top M^{-1} p \;, \tag{15}$$

where $A = A^\top \in \mathbb{R}^{n \times n}$ satisfies the linear matrix inequality (LMI)

$$A + \nabla_r^2 c^i(x^*) \xi_i^* + \left(\nabla_r c(x^*) \nabla_r c^\top(x^*)\right)^j \bar{\xi}_j^* > 0 \tag{16}$$

for some scalars ξ_i^* and $\bar{\xi}^*$, and with

$$a = -\nabla_r c^i(x^*) \xi_i^* \;. \tag{17}$$

Since the kinetic energy is left unchanged, we have $D_\lambda(H_{\mathrm{d}}) = D_\lambda(H)$, so (11) is trivially satisfied. Since

$$D_{H_{\mathrm{a}}} = H_d - H \in \operatorname{span}\left\{ \begin{pmatrix} 0_n \\ I_n \end{pmatrix} \right\} \;,$$

equation (10) is solvable on account of full actuation. Thus, the closed-loop is passive with storage function (15). To show stability, it suffices to prove (14).

Next, we construct the Lagrange function

$$L(x, \xi) = H_{\mathrm{d}}(x) + c^i(x) \xi_i$$

with Lagrange multipliers ξ_i. The first-order stationarity condition gives

$$a + A(r - r^*) + \nabla_r c^i(x) \xi_i = 0 \;,\; M^{-1} p = 0 \;,\; c^i(x) = 0 \;,$$

which are solved by (13) and $\xi_i = \xi_i^*$ if we set a as in (17).

The second-order sufficient condition takes the form (Berstekas, 1996, p. 68)

$$z^\top \left(A + \nabla_r^2 c^i(x^*) \xi_i^*\right) z > 0 \tag{18}$$

for all $z \in T_{x^*}\Gamma$, i.e., for all $z \in \mathbb{R}^n$ such that $\nabla_r (c^i)^\top (x^*) z = 0$, $i = 1, \ldots, k$.

It remains to show that the condition (18) is satisfied whenever (16) holds. LMI (16) can be equivalently written as an inequality involving quadratic form

$$\langle y, \left(A + \nabla_r^2 c^i(x^*) \xi_i^* + \left(\nabla_r c(x^*) \nabla_r c^\top(x^*)\right)^j \bar{\xi}_j^*\right) y \rangle > 0$$

which must hold for all $y \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Chosing $y \in T_{\underline{x}^*}$ we recover (18) while the converse, i.e., the existence of $\bar{\xi}^*$ follows from the Finsler theorem (Bellman, 1970). □

The LMI (16) can always be solved by setting $\xi_i = 0$, $\bar{\xi}_j = 0$ and choosing A any positive definite matrix. However, the resulting controller can be greatly simplified by carefully solving the LMI.

Below, we give some intuition on how to choose the coefficients in equations (15) and (16). To do this we have to consider in some more detail the class of systems under study.

A typical mechanical system can be modelled as a set of point masses with some constraints imposed on them. In absence of electro-magnetic field, the potential energy of such a system is described as a sum of gravitational potential energies of the respective masses and is hence a linear function of the system's coordinates r.

We consider only non-holonomic, i.e., geometrical constraints (note that the integrable kinematic constraints can be considered within the same framework). The most typical geometric constraints are

- The *linear* constraints, i.e., the constraints of the form $a_i r^i + b = 0$, where $a_i, b \in \mathbb{R}$. These constraints effectively eliminate some degreees of freedom of the system fixing the values of the respective coordinates.

- The *quadratic* constraints, i.e., those of the form $\|\alpha_i r^i\|^2 + \beta = 0$, $\alpha_i, \beta \in \mathbb{R}$. These constraints state that the respective point masses must maintain fixed distances from each other.

The equilibrium stabilization problem consists in finding a Hamiltonian function $H_d(x)$ whose restriction to $\mathcal{M}_\Gamma$ attains its minimum value at $x = x^*$, see Eq. (13).

The kinetic energy is a positive-definite quadratic function of q and hence attains its minimum at $q^* = 0$. For the potential energy there are two options: either the configuration manifold is convex, i.e., there is exists a vector $\xi^* \in \mathbb{R}^k$, $\xi_i^* > 0$ such that the weighted sum of Hessians of constraints $c^i(r)$ is positive definite, $\nabla_r^2 c^i(x^*)\xi_i^* > 0$, or the Hamiltonian H_d is a convex function of r. The first case occurs when there are quadratic constraints imposed on the system. In this case, the Hamiltonian function can be chosen to be linear, i.e., the matrix A can be set to zero. In the second case, the existence of a global minimum is guaranteed by choosing $A > 0$. A detailed analysis of this issue will be presented in an extended version of this paper which will be published elsewhere.

Remark 9. The controller obtained from the matching equation (10) with H_d as in (15) provides Lyapunov stability only. As usual, asymptotic stability can then be achieved by adding proper damping.

For underactuated systems, the problem can be solved by searching first a set of $s^i(x)$, $i = 1, \ldots, m+k$, such that

$$\operatorname{span}\{\Omega^{-1}(ds^i)\} = \operatorname{span}\{D_\lambda\} \oplus \operatorname{span}\{D_g\} \ . \tag{19}$$

By setting the desired Hamiltonian as $H_\mathrm{d}(x) = H(x) + f(s(x))$, it is ensured that H_d is assignable for any differentiable $f : \mathbb{R}^{m+k} \to \mathbb{R}$. Then f is chosen such that (14) holds.

The described approach has a number of advantages compared to solving the equilibrium stabilization problem in local coordinates (Ortega et al., 2002; Ortega and García-Canseco, 2004; Acosta et al., 2005). In particular, one needs to solve a simple quadratic program instead of a partial differential equation. Furthermore, the obtained control is expressed in global coordinates, hence, there are no singularities. Finally, it turns out that an implicit Hamiltonian system is easier to discretize as the Hamiltonian function written in global coordinates is separable. This fact can be used to design an effective integration scheme (Castaños et al., 2015).

3.3 The simple actuated pendulum

Suppose we want to stabilize the point $x^* = (0\ l\ 0\ 0)^\top$ (the upright vertical position), which clearly satisfies (13). A solution set for the LMI (16) is

$$A = 0 \ , \ \xi_1^* = \frac{m_1\bar{g}}{l} \ , \ \bar{\xi}_1^* = 0 \ ,$$

i.e., $\nabla_r^2 c^1(x^*) = I_2 > 0$. This gives

$$a^\top = -(0\ l)\cdot\frac{m_1\bar{g}}{l} = -(0\ m_1\bar{g})$$

and

$$H_\mathrm{d}(x) = -m_1\bar{g}r^y + \frac{1}{2m_1}\|p\|^2 \ .$$

The matching equation (10) takes the specific form

$$\begin{pmatrix} 0 \\ 2m_1\bar{g} \end{pmatrix} = -\begin{pmatrix} r^x \\ r^y \end{pmatrix}\mu + \frac{1}{l^2}\begin{pmatrix} -r^y \\ r^x \end{pmatrix}\hat{u} \ . \tag{20}$$

The solution is simply

$$\mu = -2m_1\bar{g}\frac{r^y}{l^2} \quad \text{and} \quad \hat{u} = 2m_1\bar{g}r^x \ .$$

A local coordinate chart for Γ is $\theta \mapsto (l\sin\theta\ l\cos\theta)^\top$ with $\theta \in (-\pi, \pi)$. In local coordinates, the control takes the form

$$\hat{u} = 2m_1\bar{g}\sin\theta \ .$$

Since it was constructed using global coordinates, the controller does not exhibit undesirable phenomena such as *unwinding* (Chaturvedi et al., 2011). Moreover, $\hat{u}$ is continuous on Γ at $\theta = \pi$, the point which is not covered by the coordinate chart.

3.4 The pendulum on a cart

It can be verified that

$$s^1(x) = r^{1_y} \ , \quad s^2(x) = \frac{1}{2}\|r^2 - r^1\|^2 \quad \text{and} \quad s^3(x) = r^{1_x}$$

satisfy (19). Given the constraints $c^1(x) = c^2(x) = 0$, it is clear that the only Hamiltonian functions of interest are of the form

$$H_\mathrm{d}(x) = H(x) + f(s^3(x)) = \frac{1}{2m_1}\|p_1\|^2 + \frac{1}{2m_2}\|p_2\|^2 + m_2\bar{g}\cdot r^{2_y} + f(r^{1_x}) \ ,$$

and that the stabilizable equilibria have the structure

$$x^* = \begin{pmatrix} r_* \\ 0 \end{pmatrix} \ ,$$

where $r^* \in \left\{ \left(r_*^{1_x}\ 0\ r_*^{1_x}\ -l\right)^\top \mid r_*^{1_x} \in \mathbb{R} \right\}$.

Condition (14) is indeed satisfied with the choice $f(r^{1_x}) = \frac{1}{2}(r^{1_x} - r_*^{1_x})^2$. The matching equation is then

$$\begin{pmatrix} r_*^{1_x} - r^{1_x} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & r^{1_x} - r^{2_x} \\ 1 & r^{1_y} - r^{2_y} \\ 0 & r^{2_x} - r^{1_x} \\ 0 & r^{2_y} - r^{1_y} \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\hat{u} \ . \tag{21}$$

The solution is simply $\mu_1 = \mu_2 = 0$ and $\hat{u} = r_*^{1_x} - r^{1_x}$.

4. CONCLUSIONS

In global coordinates, the defining functions (the Hamiltonian and the constraints) of many mechanical systems of interest are quadratic and convex. This representation proves to be useful in an energy shaping scenario, where the control problem turns out to be a simple quadratic programming problem instead of the usual problem of finding the solution of a partial differential equation. Another advantage of computing the closed-loop energy (or Lyapunov) function is that the resulting controller does not exhibit undesired phenomena such as winding.

It is worth noting, however, that once the closed-loop Hamiltonian has been obtained, computing the control is simpler in local coordinates. Thus, the results of Castaños et al. (2013) can be used in a mixed approach in which H_d

or V are computed in global coordinates and the actual control is computed using an explicit representation.

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