

# Output Feedback Sliding-Mode Control with Unmatched Disturbances, an ISS Approach

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## 1 Introduction

Throughout the history of control there has existed an interest to understand the behavior of dynamic systems with different inputs which can be perturbations, noise, or control laws. Many efforts have been dedicated to answer questions such as what kind of inputs will let a stable system maintain such property, and how to characterize this stability. In particular, the study of these topics for non linear systems has aroused a lot of interest. At the end of the eighties the first notions that answered this questions appeared, and were gathered under the name of input to state stability (ISS). This theory established conditions under which a norm (usually Euclidean or supremum) of the states is eventually bounded by the norm of its inputs, and tends to zero when the inputs are absent [1].

Many advances in the ISS theory were made in the following decades, for example, establishing the sufficient and necessary conditions to characterize a system as ISS [1, 2]. Also, the interconnection of systems has been a central subject in many works, resulting in some useful and widely known theorems such as a Lyapunov-based nonlinear small gain theorem [3], or a small gain theorem for systems with mixed ISS characterizations [4]. On the other hand, many Lyapunov approaches have been developed to facilitate the ISS analysis by means of Lyapunov functions [5]. These advances have led to the discovery of many applications to the ISS theory.

Recently, the ISS theory has incorporated a new concept: the integral input to state stability (iISS), allowing inputs to be bounded by an integral norm and states by a supremum one [6]. This new concept enriches the ISS theory by allowing to characterize the stability of a broader class of systems that could not be characterized as ISS such as, the conventional sliding-mode controller with constant gain. This approach, however, is still largely unexplored, and its implementation can be complicated. A methodology that has proven to facilitate the ISS and iISS analysis is through the use of weighted homogeneity [7], which is very convenient for some sliding-mode algorithms with an homogeneous nature. The disadvantage of this approach is that, although ISS can be established on homogeneous grounds it is still impossible to calculate an iISS or an ISS gain.

The sliding-mode controllers have gained a great deal of popularity since their introduction, more than 30 years ago. Much of this popularity is due to their robustness against matched disturbances in the knowledge of an upper bound. They can provide finite-time convergence to a sliding surface specifically designed for the desired dynamic behavior of a system [8].

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The downside in the use of the sliding-mode controllers is that their robustness is severely compromised when a bound for the disturbance acting over the system cannot be known, or when they are unmatched to the control input. Some work has been done in this direction [9, 10], offering solutions to the unmatched disturbance problem. Another disadvantage of the systems governed by sliding modes is that they may lose their robustness properties when directly connected to other systems, for example, observers.

In the early eighties, regular forms were introduced [11, 12], and have been widely used as a way of simplifying the selection of sliding manifolds and control laws. These forms offer a simple visualization of system properties, dividing the system in two: a subsystem that contains the control and another subsystem that does not. It is worth mentioning that in order to implement a sliding-mode control on a system in regular form, the complete state must be measured. In the nineties, the backstepping theory developed, and a similar two stage technique that also requires complete knowledge of the state, appeared.

The normal form that appears in [13] follows the same idea as the regular forms, in the sense that it separates the system into the part that contains the control input, and the part that does not, but taking into account the case when the system has an output.

A good approach to solving the output feedback problem using a sliding mode controller, combining a norm observer that estimates the norms of the uncertain states, and a time-varying high gain observer that estimates the output derivatives, can be found in [14]. The goal of the mentioned work is to achieve global tracking of an arbitrary relative degree output, by turning the problem into a regulation one. This work considers a general non-linear system that can be transformed to the normal form. However, the development needs that the zero dynamics are stable a priori and does not consider external disturbances of any kind. Another similar approach that guarantees global stability by combining a high gain observer that takes the output derivative estimation error to a vicinity of the origin, and a robust exact differentiator that takes the error to zero, through a hybrid design that switches between the two, was presented in [15]. This work also avoids the peaking phenomena but the same assumptions on the stability of the zero dynamics are required and, again, the uncertainties considered are only state dependent and not external. Both of these results use ISS tools to guarantee the stability, proving this property for the tracking error dynamics, although the ISS behaviour of the zero dynamics with respect to the rest of the the state is only assumed to exist. Another hybrid scheme, similar to the work mentioned above, was introduced in [16] for SISO systems, this time switching between a lead filter when the error is far from the origin, and the robust exact differentiator when the error is inside a neighborhood of the origin. This approach was recently extended to MIMO systems in [17]. In both of the latter papers ISS tools are also used to prove the stability of the tracking and the estimation error dynamics. Once these two new dynamic systems are proved ISS with respect to each other, a small gain analysis is used to prove the stability of the closed loop. Unfortunately, the assumptions on the stability of the zero dynamics are maintained and, even though these two works consider external disturbances, the scheme only supports matched ones and requires the knowledge of an upper bound for them. These known results establish very useful methodologies to overcome the relative degree problem by achieving an exact estimation of the output derivatives in order to construct a relative degree one sliding variable, but leave open the problem of controlling a possibly unstable zero dynamics and the handling of unmatched disturbances. The central focus of the present work is to solve the problem of virtually controlling a possibly unstable zero dynamics, affected by unmatched disturbances, for a system in a special case of the normal form and, at the same time, ensuring that the complete state trajectories of the system, also affected by matched disturbances, remain in a neighborhood of the origin, using a conventional sliding-mode controller.

One of the main contributions of [9] is the introduction of an output-based regular form for

a system with an output of relative degree one. This output-based regular form was generalized for arbitrary relative degree in [18]. In the present work we introduce a state transformation capable of transforming the state into an output normal form, in which the input of the zero dynamics is the measured output only, and not a function of the output and its derivatives, as usual. It is also shown that the new subsystems are controllable and observable if the original system.

This work is motivated by the interest of studying the conditions under which a system governed by sliding-mode controllers can show an ISS behavior, and the characterization of this property. This would allow to calculate ISS gains of the system with respect to matched and unmatched disturbances with unknown bounds, and also be useful in the analysis of interconnections with other systems. In this paper we perform such analysis for a system governed by a conventional sliding-mode controller. The main contribution of this article is the ISS Lyapunov-based analysis of a system with an arbitrary relative degree output and unstable zero dynamics, governed by a conventional sliding-mode controller with an added linear term. This analysis leads to the introduction of a control law, and conditions for its gains that guarantee global convergence to a neighborhood of the origin of the trajectories of a system with unmatched disturbances.

The development of the work is presented in various stages and the paper is organized as follows: Section 2 presents a simple motivational example that shows the advantages of adding the above linear term to a discontinuous controller. In Section 4 an output-based normal form is introduced, which is a special case of the classical normal form. A state transformation that takes a linear system, without loss of generality, to this output normal form is provided. In section 5 a controllable and observable reduced order system, which contains the unmeasurable part of the state is presented. Section 6 proposes, for the unmeasurable state, i.e., the zero dynamics, the construction of an observer and a dynamic virtual control law that stabilizes it. The stability proof is done by an ISS-Lyapunov based analysis. In Section 7 a sliding surface and a discontinuous control law with an added linear term that enforces the sliding mode are defined, which ensures the global stability of a system of arbitrary relative degree, with only output information available, and matched and unmatched disturbances. The mentioned combination of a discontinuous control and a linear term is the key feature that allows to ensure the global convergence, as is shown in the following motivational example of the behavior of a system in the presence of a growing input.

## 2 Motivational Example

It was mentioned in the previous section that the main idea behind studying a system governed by a sliding-mode controller via an ISS approach is to add a linear term to this controller. In order to illustrate the advantages of adding such term, the following motivational example is presented.

Consider the system

$$\dot{x} = u + w, \quad x(0) = 0, \tag{1}$$

where  $x$  is the state variable,  $u$  is the control input, and  $w$  is a growing disturbance.

Figure 1 illustrates the behavior of the trajectories of system (1) when the control input is defined as  $u = -x$  (continuous line),  $u = -\text{sign}(x)$  (dotted line), and  $u = -x - \text{sign}(x)$  (dashed line), as the input  $w$  grows. When the control is simply a linear function of the state, the ultimate bound on the state starts to grow as soon as the disturbance is different from zero. In the second case, when the control is only a discontinuous function of the state, it is capable of forcing the trajectories to the origin for some values of the perturbation, but once it

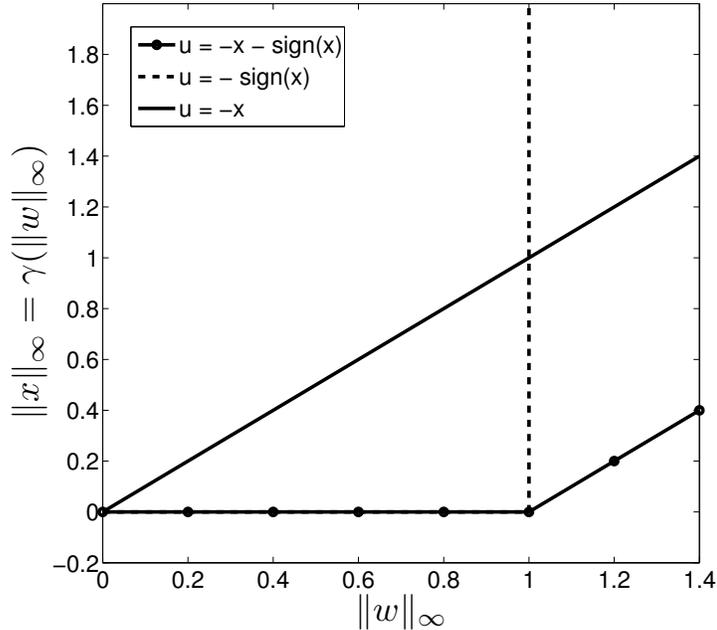


Figure 1: Behavior of the trajectories of  $x$  in the presence of a growing input  $w$  with different control laws.

surpasses a certain level (equal to one in our example), the trajectories growing unboundedly. On the other hand, when the conventional sliding-mode controller is combined with the linear term, the trajectories can remain at the origin for some values of the disturbance, and then the ultimate bounded grows with the perturbation.

### 3 Problem Statement

Consider a system

$$\begin{aligned} \dot{x} &= Ax + Bv + Dw \\ y &= Cx, \end{aligned} \tag{2}$$

where  $x \in \mathbb{R}^n$  is the state,  $v \in \mathbb{R}$  is the control input,  $y \in \mathbb{R}$  is the measured output, and  $w \in \mathbb{R}^q$  is a bounded disturbance. For simplicity, the SISO case is considered, but all the calculations can also be done for the MIMO case. It is clear that if system (2) is of dimension  $n > 1$ , being  $y$  of dimension one, there is part of the state that cannot be recovered by purely algebraic means.

For system (2) the following assumption is made:

**Assumption 1** *The output  $y$  has a relative degree  $r$  with respect to the control input  $u$ , and  $r_w$  with respect to the disturbance  $w$ , and they satisfy  $r \leq r_w \leq n$ .*

This assumption indicates that system (2) may have a zero dynamics, i.e. some internal dynamics that are present when the output  $y$  and its  $r$  successive derivatives are equal to zero. It also indicates that system (2) has matched disturbances (those that appear in the state

equations associated to the control input), and also unmatched disturbances that affect only the mentioned zero dynamics and no other state equations in the system.

This paper addresses the problem of finding a control law, and conditions for its gains, such that the trajectories of system (2) can be globally and asymptotically taken to the origin, despite the magnitude of the initial conditions, in absence of disturbances. In presence of disturbances the trajectories should globally converge to a vicinity of the origin. This control law is a conventional sliding mode controller with an added linear term. This choice of controller allows to explicitly calculate gain functions relating the magnitude of the disturbances to the magnitude of the system states. The design will be carried out in various stages, presenting various preliminary results in different lemmas, which will be condensed into one main theorem at the end of the paper.

## 4 Output Normal Form and Linear Transformations

The utility of using state transformations that allow to represent system states in a particular form has been exploited throughout the history of system's theory. In Section 1 two of these forms were mentioned: the classical regular form [12], which facilitates the sliding surface design, and the normal form [13], which allows to clearly visualize the zero dynamics of a system. In this section, a transformation that permits, without loss of generality, to represent a system in a form which will be referred to as Output Normal Form (ONF) is introduced. This form inherits some properties of the aforementioned forms, in the sense that it separates the system dynamics in two: the part that represents the zero dynamics, and the rest. In [18] a linear version of the transformation of [13] to a normal form was introduced, which for a linear system (2) is

$$\begin{aligned}
\dot{\bar{\xi}} &= \bar{A}_\xi \bar{\xi} + \bar{E}_\xi \bar{z} + \bar{D}_\xi w_\xi \\
\dot{\bar{z}}_1 &= \bar{z}_2 \\
&\vdots \\
\dot{\bar{z}}_{r-1} &= \bar{z}_r \\
\dot{\bar{z}}_r &= \bar{E}_z \bar{\xi} + \bar{A}_z \bar{z} + \bar{D}_z w_z + u \\
y &= \bar{z}_1
\end{aligned} \tag{3}$$

with  $\xi \in \mathbb{R}^{(n-r)}$ .

**Remark 1** *In the absence of disturbances ( $w_\xi = 0$ ,  $w_z = 0$ ), the complete substate  $\bar{z}$  can be taken exactly to zero, and the  $\bar{\xi}$ -subsystem becomes  $\dot{\bar{\xi}} = \bar{A}_\xi \bar{\xi}$ , which represents the zero dynamics of the complete system.*

The ONF considered for this work is

$$\dot{\xi} = A_\xi \xi + E_{\xi 1} z_1 + D_\xi w_\xi \tag{4}$$

$$\dot{z}_1 = z_2 \tag{5}$$

$\vdots$

$$\dot{z}_{r-1} = z_r$$

$$\dot{z}_r = A_z z + E_z \xi + u + D_z w_z$$

$$y = z_1,$$

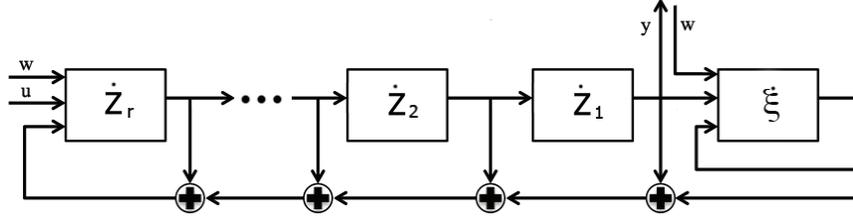


Figure 2: Output normal form.

and is illustrated in a block diagram in Figure 2. It is easy to observe that the difference between (3) and (4)-(5) is that, in the latter, the zero dynamics, represented by  $\xi$ , is driven by the output  $z_1$  only, instead of the output itself and its derivatives. This slight variation between the classical normal form, and the ONF, makes a difference in the controllability of the zero dynamics: in the latter,  $\xi$  can be controlled using the output only and, in the former,  $\xi$  is controlled through the output and its first  $r - 1$  derivatives. The following proposition introduces a transformation that takes a linear system to its ONF.

**Proposition 1** *A coordinate transformation with invertible  $T$  that brings a linear system (3), to the form (4)-(5) is*

$$\begin{bmatrix} \xi \\ z \end{bmatrix} = T \begin{bmatrix} \tilde{\xi} \\ \tilde{z} \end{bmatrix}$$

with

$$T = \begin{bmatrix} I_{n-r} & -[\bar{E}_{\xi 2} \ \cdots \ \bar{E}_{\xi r} \ 0_{(n-r) \times 1}] \\ 0_{r \times (n-r)} & I_r \end{bmatrix}$$

and  $\bar{E}_{\xi} = [\bar{E}_{\xi 1} \ \cdots \ \bar{E}_{\xi r}]$ .

**Remark 2** *Since this transformation can be applied to any linear system with relative degree  $r$ , without loss of generality, the rest of the development of this paper will be done for an ONF.*

## 5 Controllability and Observability of the Reduced Order System

It was mentioned in Section 1 that the control design presented in this work is a multi-staged one, which finally leads to the introduction of a control law that brings the trajectories of (2) to a neighborhood of the origin. In section 4 it was shown that any linear system can be taken to its output-normal form so, without loss of generality, we will perform the complete analysis for a system in such form. The first stage of the design will focus on subsystem (4), which contains the unmatched disturbance, and whose state is unmeasurable. The controllability of the zero dynamics has been a classical preoccupation when using block forms such as the Regular Form. A proof of controllability for the pairs  $(A_{\xi}, E_{\xi})$  of (3) appears in [12]. In this section a controllability proof for the pair  $(A_{\xi}, E_{\xi 1})$  is provided. Since this work deals with systems of which only output information is available, observability is also an issue to be taken into account. For this, it is shown how to construct not only a controllable, but an observable reduced order system, composed of the unmeasurable state  $\xi$ , and a virtual output which will also be defined.

Recall that a system (2) is controllable iff

$$\text{rank} [\lambda I - A \quad B] = n,$$

for all  $\lambda \in \mathbb{C}$ , and observable if

$$\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n,$$

for all  $\lambda \in \mathbb{C}$ . For the output normal form (4)-(5) this can be written as

$$\text{rank} \left( \lambda I_n - \begin{bmatrix} A_\xi & E_\xi & 0_{(n-r) \times (r-1)} & 0_{(n-r) \times 1} \\ 0_{(r-1) \times (n-r)} & 0_{(r-1) \times 1} & I_{(r-1)} & 0_{(r-1) \times 1} \\ E_z & a_{z1} & A_{zr} & 1 \end{bmatrix} \right) = n$$

where  $[a_{z1} \quad A_{zr}] = A_z$  with  $A_{zr} \in \mathbb{R}^{1 \times (r-1)}$ .

Note that the last column is composed of zeros except for the last element. This makes the last row linearly independent of the rest, so it can be discarded of the analysis, along with the remaining zero elements, leaving the matrix as

$$\text{rank} \left( \lambda I_{(n-1)} - \begin{bmatrix} A_\xi & E_\xi & 0_{(n-r) \times (r-1)} \\ 0_{(r-1) \times (n-r)} & 0_{(r-1) \times 1} & I_{(r-1)} \end{bmatrix} \right) = n - 1.$$

The same happens with the last  $r - 1$  rows and columns of the above matrix, so the rank condition becomes

$$\text{rank} [\lambda I_{(n-r)} - A_\xi \quad -E_\xi] = n - r.$$

A similar procedure can be carried out for the observability matrix:

$$\text{rank} \left( \lambda I_n - \begin{bmatrix} A_\xi & E_\xi & 0_{(n-r) \times (r-1)} \\ 0_{(r-1) \times (n-r)} & 0_{(r-1) \times 1} & I_{(r-1)} \\ E_z & a_{z1} & A_{zr} \\ 0_{1 \times (n-r)} & 1 & 0_{1 \times (r-1)} \end{bmatrix} \right) = n,$$

so, analogously, the rank condition becomes

$$\text{rank} \begin{bmatrix} \lambda I_{(n-r)} - A_\xi \\ -E_z \end{bmatrix} = n - r,$$

which proves the following Lemma:

**Lemma 1** *If the pair  $(A, B)$  of (2) is controllable, and the pair  $(A, C)$  of (2) is observable, then  $(A_\xi, E_\xi)$  is controllable and  $(A_\xi, E_z)$  is observable.*

To relate the observability of the pair  $(A_\xi, E_z)$ , with subsystem (4), it is necessary to recover the term  $E_z \xi$ . If  $y$  is a noiseless output, one can take its derivatives until the  $r^{\text{th}}$  order, and define a virtual output as

$$y_v := y^{(r)} - A_z [y \quad \dots \quad y^{(r-1)}]^\top - u$$

which, along with (4), forms the reduced order system

$$\begin{aligned} \dot{\xi} &= A_\xi \xi + E_\xi z_1 + D_\xi w_\xi \\ y_v &= E_z \xi + D_z w_z \end{aligned} \tag{6}$$

**Remark 3** In the last few lines it is assumed that  $r$  derivatives of the output  $y$ , i.e. the variables  $z_1, \dots, z_r$ , can be obtained, in order to construct the virtual output  $y_v$ . It is not the goal of this paper to explore the differentiator techniques but, in order to obtain the necessary estimates, various methods can be implemented. For example, the robust exact differentiator of [19] if a known bound of the  $(r+1)$ th derivative of  $y$  is available, the modification that provides uniformity with respect to the initial conditions of [20], a hybrid global approach as the ones that appear in [15, 16, 17], or the recent result [21] which presents a global exact differentiator based on higher-order sliding modes and dynamic gains, amongst others.

## 6 Virtual Control and Observer Design for the Reduced Order System

Since the controllability and observability of (6) have already been proved in Section 5, an observer and virtual control can be designed for it. Naturally,  $z_1$  (the measured output) is used as a virtual control for  $\xi$ . This control signal will be driven by the dynamics of an observer for the unmeasurable state  $\xi$ . The procedure described in the following lines has already been developed in detail in [22] and [23] so only a short overview of the results and necessary design steps will be performed in this section.

First, an auxiliary variable  $\eta \in \mathbb{R}^{n-r}$  is defined as

$$\eta := \beta - B_\eta y^{(r-1)},$$

with

$$\dot{\beta} = A_\eta (\beta - B_\eta y^{(r-1)}) + B_\eta \left( A_z [y \ \dots \ y^{(r-1)}]^\top + u \right),$$

where  $A_\eta \in \mathbb{R}^{(n-r) \times (n-r)}$ , and  $B_\eta \in \mathbb{R}^{(n-r) \times 1}$  are parameters that can be chosen appropriately by the designer. The dynamics of  $\eta$  recover an observer-like form, that has  $y_v$  as an input:

$$\begin{aligned} \dot{\eta} &:= A_\eta \eta - B_\eta y_v \\ &= A_\eta \eta + E_\eta \xi + D_\eta w_\eta, \end{aligned} \tag{7}$$

where  $E_\eta = -B_\eta E_z$ ,  $D_\eta = -B_\eta D_z$ , and  $w_\eta = w_z$ .

Second, a scalar signal  $\phi_1 = \phi_1(z_1, \eta)$ , with  $K_v \in \mathbb{R}^{1 \times (n-r)}$  as design parameter, is constructed as

$$\phi_1 = z_1 - K_v \eta. \tag{8}$$

Substituting (8) in (4) one obtains the augmented system

$$\begin{aligned} \dot{\xi} &= A_\xi \xi + B_\xi \eta + D_\xi w_\xi + E_\eta \phi_1 \\ \dot{\eta} &= A_\eta \eta + E_\eta \xi + D_\eta w_\eta, \end{aligned} \tag{9}$$

where  $B_\xi = E_\xi K_v$ .

**Remark 4** The value of the constants  $B_\xi$ ,  $E_\eta$  and  $A_\eta$  of (9) are determined when the dynamics of  $\eta$ , and the gain  $K_v$  of the virtual control are designed. These parameters should be selected such that the nominal part of (9) ( $\|\phi_1\| = \|w_\xi\| = \|w_\eta\| = 0$ ) is globally asymptotically stable. This is possible due to the controllability and observability of (4)-(5), and can be done, among other linear methods, by pole allocation, LQR, or  $H_\infty$ .

## 7 Control design and ISS analysis

In the previous section, a virtual control  $K_v\eta$  was defined for the zero dynamics (4). This control signal must be followed by the output  $y = z_1$ . In this section a sliding surface will be defined such that when made zero it will ensure the needed tracking, and the convergence of the rest of the states to a neighborhood of the origin. A control law that ensures the sliding mode will also be proposed, as well as sufficient conditions for its gains to guarantee the convergence.

Consider the new set of coordinates  $\rho \in \mathbb{R}^{2(n-r)}$ ,  $\phi \in \mathbb{R}^{(r-1)}$  and  $\sigma \in \mathbb{R}$ , defined as

$$\begin{aligned}\rho &:= \begin{bmatrix} \xi \\ \eta \end{bmatrix} \\ \phi_1 &:= -K_v\eta + z_1 \\ \phi_2 &:= z_2 \\ &\vdots \\ \phi_{r-1} &:= z_{r-1} \\ \sigma &:= z_r - k_{r-1}z_{r-1} - \dots - k_2z_2 - k_1(z_1 - K_v\eta),\end{aligned}\tag{10}$$

where the new scalar constants  $k_1, \dots, k_{r-1}$  are parameters that must be chosen by the designer, following some conditions that will be established in the following paragraphs.

It should be noted that if the new variable  $\phi_1$  is taken to zero, the virtual control design,  $z_1 = K_v\eta$ , of section 6 will be reached. Also, that the variable  $\sigma$  is of relative degree one with respect to the control input, and that it depends on the states  $z_1, \dots, z_r$  which are assumed to be available through the implementation of a differentiation technique such as the ones mentioned in remark 3.

System (9) can be taken to the new set of coordinates  $\rho, \phi, \sigma$  by a linear transformation

$$\begin{bmatrix} \rho \\ \phi \\ \sigma \end{bmatrix} = T \begin{bmatrix} \xi \\ \eta \\ z_1 \\ [z_2 \ \dots \ z_{r-1}]^\top \\ z_r \end{bmatrix},$$

with

$$T = \begin{bmatrix} I_{(n-r)} & 0 & 0 & 0 & 0 \\ 0 & I_{(n-r)} & 0 & 0 & 0 \\ 0 & -K_v & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{(r-2)} & 0 \\ 0 & -k_1K_v & k_1 & [k_2 \ \dots \ k_{r-1}] & 1 \end{bmatrix}.$$

This leads to

$$\dot{\rho} = A_\rho\rho + E_\rho\phi + D_\rho w_\rho \tag{11}$$

$$\dot{\phi} = A_\phi\phi + E_\phi\rho + F_\phi\sigma \tag{12}$$

$$\dot{\sigma} = E_\sigma \begin{bmatrix} \rho \\ \phi \end{bmatrix} + D_\sigma w_\sigma + u, \tag{13}$$

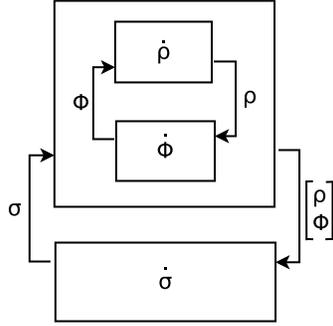


Figure 3: Feedback interconnections of (11)-(12)-(13).

where

$$A_\rho = \begin{bmatrix} A_\xi & B_\xi \\ E_\eta & A_\eta \end{bmatrix}, \quad E_\rho = \begin{bmatrix} E_{\rho 1} \\ \vdots \\ E_{\rho(n-r)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad D_\rho w_\rho = \begin{bmatrix} D_\xi w_\xi \\ D_\eta w_\eta \end{bmatrix},$$

and

$$A_\phi = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -k_1 & \dots & \dots & \dots & k_{r-1} \end{bmatrix}, \quad E_\phi = \begin{bmatrix} E_{\phi 1} \\ \vdots \\ E_{\phi r-1} \end{bmatrix}, \quad F_\phi = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Note that this transformation leaves matrix  $A_\phi$  in the canonical control form, which can be made stable by a correct choice of the constants  $k_1, \dots, k_{r-1}$ , driving the state  $\phi$  (which includes variable  $\phi_1$ ) to zero. This is achieved when  $\sigma$  is taken to zero and thus, it can be considered as the sliding variable for the system. The following development will be devoted to the stability analysis of the complete system in stages, using an ISS approach.

It can be easily seen that subsystems (11) and (12) are connected in a feedback form. Also, that if these two systems are viewed as a single one, it would also be in a feedback connection with (13), as shown in Figure 3. In order to prove the stability of (9), we will first verify the stability of the first feedback interconnection (11)-(12) and then, that of these subsystems with (13). Note that the system of Figure 3 represents the closed loop of the augmented system (9) with the new coordinates  $\phi$  and the sliding variable  $\sigma$ . This last two depend on the original output  $y = z_1$  and its derivatives and, by taking them to zero one can guarantee the convergence to a neighborhood of the origin of the trajectories of the original system (4)-(5).

First we will analyze subsystems (11)-(12) when  $\|\sigma\| = \|w_\rho\| = 0$ , in order to derive the conditions for the parameters  $k_i, i = 1 \dots r-1$  that make the nominal feedback interconnection asymptotically stable. By the design carried out in section 6, the nominal part of subsystem  $\rho$  is already asymptotically stable and, being a linear system, it is easy to verify its ISS properties:

Consider

$$\lambda_{\min}(P_\rho)\|\rho\|^2 \leq V_\rho(\rho) = \rho^\top P_\rho \rho \leq \lambda_{\max}(P_\rho)\|\rho\|^2,$$

where  $P_\rho A_\rho + A_\rho^\top P_\rho = -Q_\rho$  for a  $Q_\rho > 0$ .

From theorem 5.1 and Corollary 5.2 of [24], the ISS gain of an LTI system such as (11) can be calculated as

$$\gamma_\rho = \frac{2\lambda_{max}^2(P_\rho)\|E_\rho\|}{\lambda_{min}(P_\rho)\lambda_{min}(Q_\rho)}.$$

For the nominal part of  $\phi$ , that is when  $\sigma = 0$ , consider

$$\lambda_{min}(P_\phi)\|\phi\|^2 \leq V_\phi(\phi) = \phi^\top P_\phi \phi \leq \lambda_{max}(P_\phi)\|\phi\|^2$$

where  $P_\phi A_\phi + A_\phi^\top P_\phi = -Q_\phi$  for a  $Q_\phi > 0$ .

The value of  $P_\phi$  and  $Q_\phi$  and thus the value of their minimum and maximum eigenvalues, will depend on the chosen constants  $k_1$  through  $k_{r-1}$ . These must be chosen such that the following inequality holds

$$\frac{2\lambda_{max}^2(P_\phi)\|E_\phi\|}{\lambda_{min}(P_\phi)\lambda_{min}(Q_\phi)} < \frac{1}{\gamma_\rho}.$$

This guarantees that the classical Small Gain theorem condition is satisfied. With this it can be concluded that the feedback interconnection of (11) with (12) can be made stable by a correct choice of  $k_1 \dots k_{r-1}$ .

If the above condition is satisfied, then the nominal part of the following system (i.e.,  $\|\sigma\| = \|w_\rho\| = 0$ ) is asymptotically stable:

$$\begin{aligned} \begin{bmatrix} \dot{\rho} \\ \dot{\phi} \end{bmatrix} &= \begin{bmatrix} A_\rho & E_\rho \\ E_\phi & A_\phi \end{bmatrix} \begin{bmatrix} \rho \\ \phi \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \sigma + \begin{bmatrix} D_\rho \\ 0 \end{bmatrix} w_\rho \\ &= A_1 \begin{bmatrix} \rho \\ \phi \end{bmatrix} + E_1 \sigma + D_1 w_\rho, \end{aligned}$$

so, for a  $Q_1 > 0$ , the Lyapunov equation  $P_1 A_1 + A_1^\top P_1 = -Q_1$  can be satisfied with a  $P_1 = P_1^\top > 0$ , and a Lyapunov function for the above system above is

$$\lambda_{min}(P_1) \left\| \begin{bmatrix} \rho \\ \phi \end{bmatrix} \right\|^2 \leq V_1(\rho, \phi) = \begin{bmatrix} \rho \\ \phi \end{bmatrix}^\top P_1 \begin{bmatrix} \rho \\ \phi \end{bmatrix} \leq \lambda_{max}(P_1) \left\| \begin{bmatrix} \rho \\ \phi \end{bmatrix} \right\|^2.$$

For subsystem  $\sigma$  consider

$$V_\sigma = \frac{1}{2}\sigma^2.$$

Defining the control input  $u$  as the following discontinuous signal with an added linear term

$$u = -k_l \sigma - k_n \text{sign}(\sigma),$$

the derivative of  $V_\sigma$  is

$$\dot{V}_\sigma = \sigma E_\sigma \begin{bmatrix} \rho \\ \phi \end{bmatrix} + \sigma D_\sigma w_\sigma + \sigma(-k_l \sigma - k_n \text{sign}(\sigma)).$$

If the non linear gain is chosen according to the classic sliding-mode theory as

$$k_n > \|D_\sigma\| \bar{w}_\sigma,$$

where  $\|w_\sigma\| \leq \bar{w}_\sigma$ , and the constant  $\bar{w}_\sigma$  is known, then analogously as was done with  $V_\rho$  and  $V_\phi$ , the outer feedback interconnection of Figure 3 can be made stable if the gains of each of its subsystems satisfy the Small Gain condition

$$\gamma_1 \gamma_\sigma < 1,$$

which can be translated into a condition for the linear gain  $k_l$ .

**Remark 5** *Note that  $w_\sigma$  is equal to  $w_z$  in (5), which represents the matched component of the disturbance  $w$  in (2). In most of the sliding-mode literature, the knowledge of an upper bound of the matched disturbance is required for the gain design. In this work we consider the case when  $\bar{w}_\sigma$  is indeed known, and take it into account for the design, but we also consider the case when this constant is not known. In a real-life case, the designer can make an educated guess of the value of  $\bar{w}_\sigma$ , depending on the specific application and use this for the design without worrying that a miscalculation could destroy the stability achieved by the virtual control and the rest of the design, since, as will be stated in theorem 1, an ISS behaviour of the complete system with respect to the disturbance  $w$  will be present.*

The condition mentioned in the above paragraph, along with the main results that were proved before in this section, are gathered together in Theorem 1. Before its enunciation, let us recall the main variables and parameters that will be used for the establishment of the condi-

Parameter	
$K_v$	Virtual control gain, designed in section 6 such that the nominal part of (9) is globally asymptotically stable.
$E_\rho, E_\phi$ $P_\rho, P_\phi$	Feedback gain matrix of subsystem (12) into (11), and viceversa, respectively. Symmetric and positive definite matrices that are the solution to the pair of Lyapunov equations $P_\rho A_\rho + A_\rho^T P_\rho = -Q_\rho$ and $P_\phi A_\phi + A_\phi^T P_\phi = -Q_\phi$
tions:	where $A_\rho = \begin{bmatrix} A_\xi & B_\xi \\ E_\eta & A_\eta \end{bmatrix}$ and $A_\phi = \begin{bmatrix} 0_{(r-2 \times 1)} & I_{r-2} \\ -k_1 & \dots & k_{r-1} \end{bmatrix}$ are the system matrices of (11) and (12) respectively, for positive definite $Q_\rho$ and $Q_\phi$ .
$E_\sigma$	Feedback gain matrix of subsystems (11) and (12) into (13).
$P_1$	Symmetric and positive definite matrix that is the solution to the Lyapunov equation $P_1 A_1 + A_1^T P_1 = -Q_1$ , where $A_1 = \begin{bmatrix} A_\rho & E_\rho \\ E_\phi & A_\phi \end{bmatrix}$ , for a positive definite $Q_1$ .

**Remark 6** *The coordinates of (11)-(12)-(13) are defined in (10).*

**Theorem 1** *If for a linear, controllable and observable system (2), of dimension  $n$ , with an output  $y$  of relative degree  $r \leq n$  and an unknown, bounded, external input  $w$  of relative degree  $r_w$ , satisfying assumption 1, the control input is selected as*

$$u = -k_l \sigma - k_n \text{sign}(\sigma),$$

where the sliding variable is defined as

$$\sigma = z_r - k_{r-1} z_{r-1} \dots - k_2 z_2 - k_1 (z_1 - K_v \eta),$$

then, for every  $w \in L_\infty$ , where  $L_\infty$  denotes the set of all measurable locally essentially bounded functions endowed with the (essential) supremum norm  $\|w\|_\infty = \sup\{\|w(t)\|, t \geq 0\} \leq \infty$ , there exist a  $\mathcal{K}$  function  $\gamma$  and a  $\mathcal{KL}$  function  $\beta$  such that the norm of the solutions, for all  $t$  will remain in a neighborhood of the origin given by [1]

$$\|x(t, x(0), w)\| \leq \beta(\|x(0)\|, t) + \gamma(\|w\|_\infty),$$

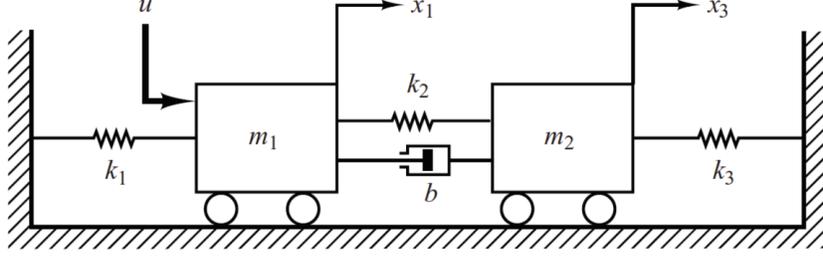


Figure 4: Damped Double Mass-Spring.

provided that the gains satisfy

$$k_l > \frac{2\|E_1\| \|E_\sigma\| \lambda_{\max}^2(P_1)}{\lambda_{\min}(P_1) \lambda_{\min}(Q_1)} \quad (14)$$

$$k_n > \|D_\sigma\| \bar{w}_\sigma, \quad (15)$$

and the parameters  $k_1, \dots, k_{r-1}$  are chosen such that the following inequality is satisfied

$$\frac{4\|E_\phi\| \|E_\rho\| \lambda_{\max}^2(P_\phi) \lambda_{\max}^2(P_\rho)}{\lambda_{\min}(P_\phi) \lambda_{\min}(P_\rho) \lambda_{\min}(Q_\phi) \lambda_{\min}(Q_\rho)} < 1. \quad (16)$$

**Remark 7** In the case when  $r < n$ , a part of the dynamics does not directly affect the output. This is the zero dynamics of the system, which can be unstable and have unknown external inputs. By defining the control law following theorem 1, the zero dynamics are virtually controlled and an ISS property with respect to unknown inputs guaranteed. When  $r = n$ , then no zero dynamics are present in the system, and the control law in theorem 1 guarantees that the trajectories converge to a the origin if a bound of the disturbances is known, and to a vicinity of the origin if this bound is unknown.

## 8 Academic example

Consider a damped double mass-spring system as the one shown in Figure 4.

The state space representation of this system can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1 + k_2}{m_1} & -\frac{b}{m_1} & \frac{k_2}{m_1} & \frac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{b}{m_2} & -\frac{k_2 + k_3}{m_2} & -\frac{b}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ m_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ w_2 \\ 0 \\ w_1 \end{bmatrix}, \quad (17)$$

$$y = [1 \ 0 \ 0 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Parameter	Value	Parameter	Value
$m_1$	0.8	$k_1$	0.4
$m_2$	0.5	$k_2$	0.5
$b$	0.6	$k_3$	0.4

Table 1: Parameters for system (17)

This system has relative degree  $r_u = 2$  and is affected by matched and unmatched disturbances. Suppose the system has the parameters shown in Table 1. For simplicity of this academic example it is assumed that all the units of the parameters are normalized so only its magnitudes are provided. A transformation

$$\begin{bmatrix} \xi \\ z \end{bmatrix} = Tx, \quad T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1.2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

takes the system to its output normal form

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0.12 & 0 \\ -1.8 & -1.2 & -0.44 & 0 \\ 0 & 0 & 0 & 1 \\ 0.62 & 1.5 & 0.67 & -0.75 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.25 \end{bmatrix} u + \begin{bmatrix} 0 \\ w_1 \\ 0 \\ w_2 \end{bmatrix}.$$

The parameters for the observer and the virtual controller that render the reduced order system stable, were chosen as

$$A_\eta = \begin{bmatrix} -1.02 & 1.167 \\ -1.96 & -1.38 \end{bmatrix}, \quad K_v = [-0.033 \quad 0.01], \quad E_\eta = \begin{bmatrix} -0.008 & -0.02 \\ 0.052 & 0.125 \end{bmatrix}.$$

Choosing the positive definite matrices  $Q_\rho = I_4$  and  $Q_\phi = 1$ , the corresponding Lyapunov equations are solved with

$$P_\rho = \begin{bmatrix} 0.9810 & -0.4998 & 0.0077 & 0.0064 \\ -0.4998 & 1.1665 & 0.0189 & 0.0263 \\ 0.0077 & 0.0189 & 0.3937 & -0.0847 \\ 0.0064 & 0.0263 & -0.0847 & 0.4855 \end{bmatrix}, \quad P_\phi = 0.25.$$

The sliding variable is defined as proposed in theorem 1 as  $\sigma = z_2 - k_1(z_1 - K_v\eta)$ . If the gain  $k_1$  is chosen as  $k_1 = 4.5$ , condition (16) is satisfied with  $\gamma_\rho\gamma_\phi = 0.47$ . The resulting matrix  $A_1$  can be verified to be stable. Choosing a positive definite matrix  $Q_1 = I_5$ , the Lyapunov equation for system (11) and (12) is solved with

$$P_1 = \begin{bmatrix} 1.0998 & -0.6245 & 0.0088 & 0.007 & 0.1040 \\ -0.6245 & 1.3876 & 0.0220 & 0.0306 & -0.0927 \\ 0.0088 & 0.0220 & 0.3940 & -0.0846 & -0.0011 \\ 0.007 & 0.0306 & -0.0846 & 0.4857 & 0.0067 \\ 0.1040 & -0.0927 & -0.0011 & 0.0067 & 0.2502 \end{bmatrix}.$$

The condition (14) establishes that the linear gain must be chosen as  $k_l > 74.2$  and condition (15) gives  $k_n > \bar{w}_\sigma$ .

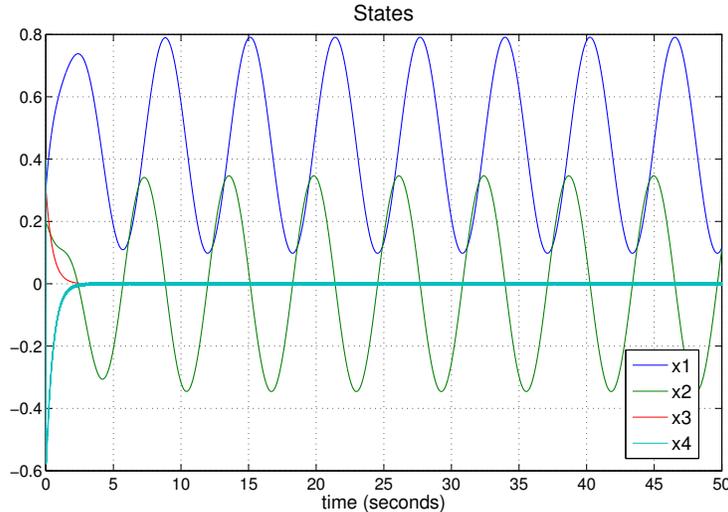


Figure 5: State trajectories of the damped double mass-spring system (17).

## 9 Simulations

The disturbance signals for the numerical simulations were chosen as  $w_1 = 1.2 + 0.6 \sin(t)$  and  $w_2 = 0.8 + 0.5 \sin(t)$ . The initial conditions were chosen as rather large for a mass-spring system, as  $x_1 = 1$  m,  $x_2 = 1.2$  m,  $x_3 = 3$  m, and  $x_4 = 2$ . With gains  $k_l = 75$  and  $k_n = 4$ , the following simulation results were obtained

Figure 5 shows how the trajectories of the system remain in a bounded neighborhood of the origin, in presence of the matched and also the unmatched disturbances. Figure 6 shows the control signal which is discontinuous for most of the simulation time, and the zoom shows the period of time where the linear term acts, before reaching the sliding-mode. Figure 7 shows the sliding surface converging to zero. The reaching time shown in the zoom coincides with the period of time where the linear control acts.

Next, the perturbations were augmented ( $w_1 = 12 + 6 \sin(t)$ ,  $w_2 = 8 + 5 \sin(t)$ ), and both the linear and the non-linear selected gains were maintained the same as before. This scenario would be problematic for a first-order sliding-mode controller with the selected gain, but Figure 8(a) shows that the combination of the linear term and the discontinuous one can still maintain the trajectories of the system in a neighborhood of the origin, showing an ISS behavior. Figure 8(b) shows the control signal. It can be seen that the control signal alternates between the discontinuous term and the linear one. This switching corresponds to the moments where the sliding-mode is lost and regained, as shown in Figure 8(c).

## 10 Conclusions

It was shown that the properties of a first-order sliding-mode controller can be combined with those of a linear term, in order to achieve enhanced global robustness of the closed loop against matched and unmatched perturbations. Moreover, it was shown that the closed loop shows an ISS behavior with respect to the matched and unmatched disturbances. A sufficient condition for the gains of this combined controller was derived using standard ISS tools such as ISS-

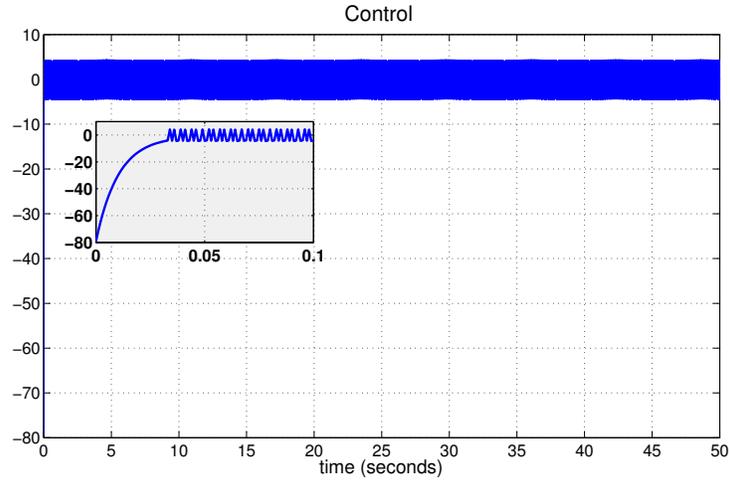


Figure 6: Control signal  $u = -75\sigma - 4 \text{sign}(\sigma)$ , and zoom to the first 0.1s which illustrates the action of the linear part of  $u$ .

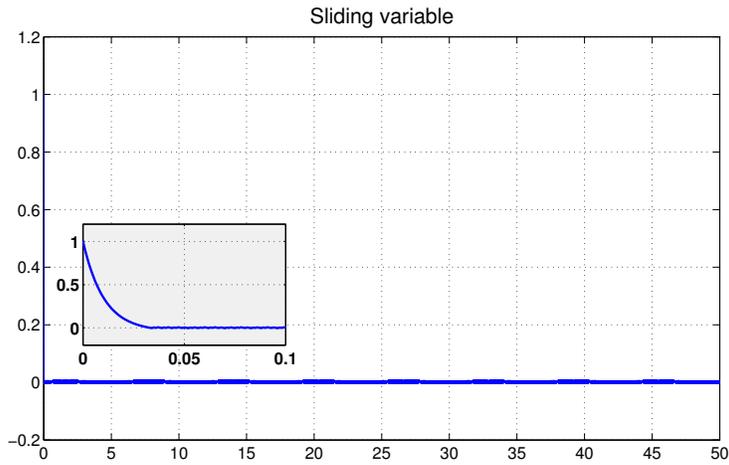


Figure 7: Trajectory of the sliding variable  $\sigma = z_2 - 4.5(z_1 + 0.033\eta_1 - 0.01\eta_2)$ , and zoom to the first 0.1s which illustrates the behavior before reaching the sliding mode.

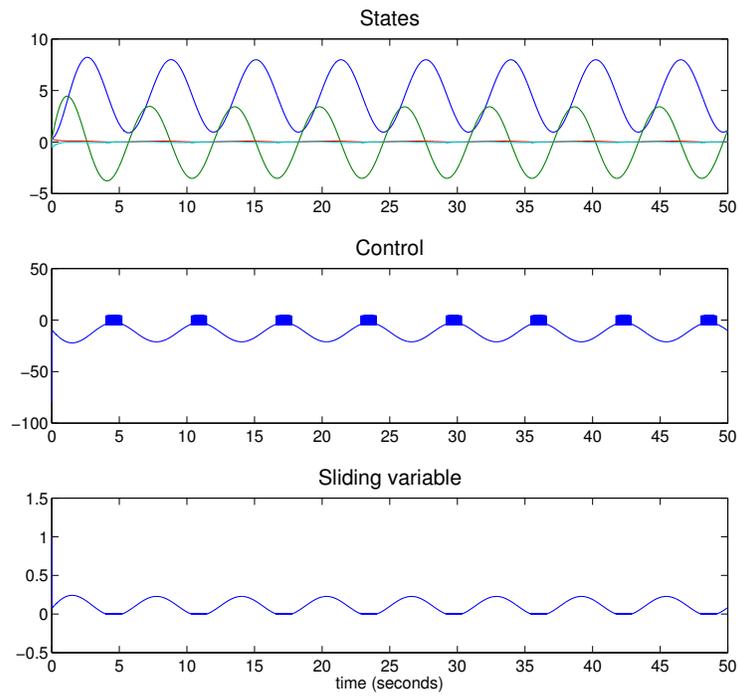


Figure 8: System's behavior with perturbations whose magnitude surpasses the magnitude of the non linear gain ( $w_1 = 12 + 6 \sin(t)$ ,  $w_2 = 8 + 5 \sin(t)$ ): (a) State trajectories of (17). (b) Control signal  $u = -75\sigma - 4 \text{sign}(\sigma)$ . (c) Trajectory of the sliding variable

Lyapunov functions and the classical Small Gain theorem. All this was done for a system for which only output information is available and has a zero dynamics. In order to complete the methodology for constructing an ISS closed loop, an observer and a virtual control for the zero dynamics were designed by deriving an observable and controllable reduced-order system from the original one. Also, in order to facilitate the visualization of the system's characteristics, a linear transformation that takes the equations to an output normal form was presented.

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