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ISS Properties of Sliding-Mode Controllers for Systems with Matched and Unmatched Disturbances

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Abstract— **In this paper we present a controller that achieves global input-to-state stabilty for a linear system of arbitrary relative degree, subjected to matched and unmatched disturbances. This controller combines the properties of a discontinous term, and a linear one, enforcing a conventional sliding mode using only partial state information. A direct and simple way of choosing the gains for this controller is also provided.**

I. INTRODUCTION

Sliding modes are well known as one of the most effective control methods to deal with systems with unknown inputs and disturbances, since they are capable of compensating them theoretically exactly when they are matched to the control input [16]. Another important feature of the SM controllers is that they provide finite-time and exact convergence of the states to a sliding surface that can be designed to ensure desirable system behavior. The drawback of this method is that it is sensitive to unmatched disturbances [6]. There are many works addressing this problem, by combining the sliding-mode theory with other control strategies. Two of the most typical combinations are with backstepping and H_∞.

Backstepping is a powerful tool to recursively design controllers for systems with unmatched disturbances, determining virtual controls that allow to track a reference in a specific channel of the system. This forces the trajectories of that channel, often subjected to unmatched disturbances, to converge to a reference [13], [8]. Combinations of backstepping and sliding modes can be found in a number of works, including [3], [4], [7]. A drawback of this theory is that it stops being useful for systems with only output information available.

H_∞ is a well known technique that characterizes controllers which ensure convergence of the states to a neighborhood of the origin. In [5] one can find a complete methodology to derive the state space formulas for finding controllers such that the H_∞ norm of the closed-loop transfer function that maps the perturbations to an error signal is less than a number $\gamma > 0$. Two representative works on the combination of H_∞ controllers and sliding modes are: [2], and [1]. In particular, in the latter, a dynamic sliding surface is designed, using only partial state information, which allows a part of the state of the system to act as a virtual H_∞ controller. This sliding mode is enforced by a first order discontinuous controller, for a system with relative degree one, and ensures local ultimate boundedness of the closed loop. In [14] an extension of the mentioned results, for a system with relative degree two, achieving local ultimate boundedness, was presented.

In this paper we present a methodology that allows to combine different virtual control strategies, to deal with unmatched disturbances, with sliding modes using only output information. This ensures the global input-to-state stability (ISS) of the complete system, which is achieved designing a relative-degree-one sliding surface, and enforcing the sliding mode with a controller that combines a discontinuous first-order term, and a linear one. This controller is also capable of compensating the matched disturbances in the surface, if they have a known upper bound. If the bound of the disturbances is unknown, the controller is able to attenuate them below an explicit gain function. The global boundedness of the trajectories is assured by an ISS [15] characterization of the controller, which derives in a set of conditions for choosing sufficient gains for it. Even though the proposed sliding surface is of relative degree one and, consequently, the sliding-mode controller is of first-order, the methodology is laid out for systems with any relative degree.

The paper is organized as follows: In Section II we introduce the notation used throughout the paper, give some definitions and detail some useful known results. Section III contains the problem statement, and some assumptions on the system properties that are necessary for the development of the paper. In Section IV we describe the virtual control design. In Section V we summarize the results in a theorem that states sufficient conditions to achieve ISS i.e. global and asymptotic stability in absence of external imputs, of the closed loop, with a conventional sliding-mode controller with an added linear term. Section VI contains an academic numerical example and simulations of the procedure described in the previous sections, for an unstable plant, and shows via simulations that the conditions found on Section V are not very far from the necessary ones. Finally, Section VII provides some conclusions to this work.

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II. PRELIMINARIES

A. Notation

In this paper we use the following notation

- $|a|$ represents the absolute value of a scalar a
- $|A|$ represents the quadratic norm of a matrix A
- If a signal is continuously differentiable n times, it is said to be $\mathscr{C}^n$
- $\lambda_{min}(A)$ and $\lambda_{max}(A)$ represent the minimum and maximum eigenvalues of a matrix A, respectevly.

B. Known results

Definition 1: [12] A system $\dot{x} = f(t,x,v)$, where x represents the states, and v represents all the external inputs of the system, including perturbations, command signals, and noises, is said to be input-to-state stable if there exists a function $\beta \in \mathscr{KL}$ and a function $\gamma \in \mathscr{K}$ such that for any initial state $x(t_0)$, and any bounded input $v(t)$, the solution $x(t)$ satisfies

$$|x(t)| \leq \beta(|x(t_0)|, t-t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} |v(\tau)|\right). \tag{1}$$

Definition 2: [11] For a system $\dot{x} = f(x,v)$, a smooth function V is said to be an ISS-Lyapunov function if V is proper, positive definite, i.e., there exists functions ψ_1, $\psi_2 \in \mathscr{K}_\infty$ such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|),$$

and there exist functions $a \in \mathscr{K}_\infty$ and $\theta \in \mathscr{K}$ such that

$$\nabla V(x) f(x,u) \leq -a(V(x)) + \theta(|v|).$$

Theorem 1: [11, Thm 3.1] If, for interconnected systems

$$\dot{x}_1 = f_1(x_1, x_2, v_1) \tag{2}$$
$$\dot{x}_2 = f_2(x_1, x_2, v_2), \tag{3}$$

there exist an ISS-Lyapunov function V_i, for the x_i subsystem, $i = \{1,2\}$, such that with functions $\phi_i \in \mathscr{K}_\infty$, χ_i, $\gamma_i \in \mathscr{K}$ the following holds:

$$V_i(x_i) \geq \max\{\chi_i(V_j(x_j)), \gamma_i(|v_i|)\} \Rightarrow$$
$$\nabla V_i(x_i) f_i(x_i, x_j, v_i) \leq -\phi_i(V_i),$$

with $j = \{2,1\}$, and

$$\chi_1(r) \circ \chi_2(r) < r \quad \forall\, r > 0, \tag{4}$$

then the interconnected system (2), (3) is ISS. This means that the zero solution of (2), (3), with $v = 0$, is globally asymptotically stable.

Corollary 1: [11] If V_i are ISS-Lyapunov functions for (2), (3), and

$$\nabla V_i(x_i) f_i(x_i, x_j, v_i) \leq -a_i(V_i(x_i)) + \theta_i^x(V_j(x_j)) + \theta_i^v(|v_i|), \tag{5}$$

for some $a_i \in \mathscr{K}_\infty$ and $\theta_i^p \in \mathscr{K}$ $(i = \{1,2\},\ p = \{x,v\})$ with

$$\theta_i^x(s) = \kappa_i a_j(s), \tag{6}$$

for some $\kappa_i > 0$, then the condition (4) is satisfied if $\kappa_1 \kappa_2 < 1$.

Theorem 1 and its Corollary, are a summary of the main results of [11]. Theorem 1 contains the formulation of the Nonlinear Small Gain Theorem in terms of ISS-Lyapunov functions for two systems in feedback interconnection. From the equations of this Theorem it is easy to see that functions $\chi_i(r)$ represent the gain functions of each of the systems with respect to their imputs v_i, and that (4) is the nonlinear representation of the classic Small Gain condition. Functions χ can be explicitly calculated from the implication in (5), and Corollary 1 offers a simpler way of verifying the satisfaction of condition (4).

C. Special Normal Forms

In [14] it was introduced a transformation that can take, without loss of generality, a linear system with relative degree r and dimension $n \geq r$, to a special normal form

$$\begin{array}{rcl} \dot{\xi} & = & A_\xi \xi + B_\xi z_1 + D_1 w_1 \\ \dot{z}_1 & = & z_2 \\ \vdots & = & \vdots \\ \dot{z}_{r-1} & = & z_r \\ \dot{z}_r & = & A_r \xi + B_r z + u + D_2 w_2 \\ y & = & z_1, \end{array} \tag{7}$$

where $y \in \mathbb{R}$ is the measured output of relative degree r with respect to the control input $u \in \mathbb{R}$, $\xi \in \mathbb{R}^{n-r}$ represents the zero dynamics when $z = 0$ and, finally, $w_1 \in \mathbb{R}^{p_1 \leq n-r}$ is a disturbance unmatched to the control input while $w_2 \in \mathbb{R}$ is a matched one. Note that the output has relative degree $r_w \geq r$ with respect to the perturbation vector $w = [w_1\, w_2]^T$.

This is a special case of the classical normal form introduced in [10].

III. PROBLEM STATEMENT

Consider the following uncertain system

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu + Dw \\ y & = & Cx. \end{array} \tag{8}$$

where $x \in R^n$ is the state vector, $w \in R^p$ is a bounded unknown input, $u \in R^m$ is the control signal and $y \in R^m$ is the measured output with relative degree r with respect to the control. For simplicity it is considered the SISO case when $m = 1$, but the calculations could be done for MIMO systems.

The goal is to obtain a control law that provides global ultimate boundedness of the closed loop, i.e. that for any initial conditions and under the influence of the disturbances, it forces the trajectories of the system to converge to a neighborhood of the origin.

Assumptions

The following assumptions are made on system (8):

a) The pair (A,B) is controllable.
b) The pair (A,C) is observable.
c) The relative degree of the output with respect to the unknown input, r_w satisfies $r \leq r_w$.
d) The unmatched component of the unknown input w is at least $\mathscr{C}^{r-2}$ and its $r-3th$ derivative is Lipschitz.
e) The matched component of the unknown input w is at least $\mathscr{C}^{r-1}$ and its $r-2th$ derivative is Lipschitz.
f) An upper bound of the matched (and unmatched) disturbance and (some of) its derivatives is known, and represented by a positive constant $\bar{w}_{\sigma k}$.

Assumption f) states that eventhough the perturbations are required to be several times differentiable, a knowledge of a bound of all their derivatives is not required.

IV. METHODOLOGY

Consider a system (8) in its special normal form (7), and perform the following steps

- Construct an auxiliary dynamic variable $\eta \in \mathbb{R}^{n-r}$ where
$$\dot{\eta} = \hat{A}\xi + \hat{B}\eta + D_\eta w_2,$$
with $\hat{A} \in \mathbb{R}^{(n-r)\times(n-r)}$, $\hat{B} \in \mathbb{R}^{(n-r)\times(n-r)}$ and $D_\eta \in \mathbb{R}^{(n-r)\times p}$
- Define a scalar variable
$$\sigma_1 = y + F\eta \tag{9}$$
with $F \in \mathbb{R}^{1\times(n-r)}$
- Substitute (9) into the state ξ of (7), and define $F_0 := B_\xi F$, to obtain the augmented system
$$\begin{aligned} \dot{\xi} &= A_0\xi + F_0\eta + D_\xi w_1 + \sigma_1 \\ \dot{\eta} &= \hat{A}\xi + \hat{B}\eta + D_\eta w_2 \end{aligned} \tag{10}$$
- Assign values to the constant matrices $\hat{A}$, $\hat{B}$, and F such that the nominal part of the augmented system ($w_i = 0$), in sliding mode ($\sigma_1 = 0$) is globally asymtotically stable (GAS).

Remark 1: The above mentioned steps are part of the methodology introduced in [1] to design a dynamic sliding surface for a system of relative degree one, with matched and unmatched disturbances. For details of this procedure please refer to the cited work.

Taking the first r succesive derivatives of signal σ_1 as a set of coordinates, one can obtain the dynamics

$$\begin{aligned} \dot{\sigma}_1 &= \sigma_2 \\ &\vdots \\ \dot{\sigma}_{r-1} &= \sigma_r \qquad (11) \\ \dot{\sigma}_r &= A_\sigma\xi + \Gamma_\sigma z + D_\sigma w_\sigma + B_\sigma\eta + u \qquad (12) \end{aligned}$$

where $A_\sigma \in \mathbb{R}^{1\times(n-r)}, B_\sigma \in \mathbb{R}^{1\times(n-r)}, D_\sigma \in \mathbb{R}^{1\times p}$, and $\Gamma_\sigma \in \mathbb{R}^{1\times r}$ are combinations of the parameters of (7), and $w_\sigma \in \mathbb{R}^{(2r-1)\times 1}$ is a perturbation vector that contains the unmatched disturbance, its $r-2$ successive derivatives, and the matched disturbance and its $r-1$ successive derivatives.

Defining the control signal u as $u := u_n + u_s$, where $u_n := -(\Gamma_\sigma z + B_\sigma\eta)$, subystems (10) and (11) form the following $2n-r$-dimensional system

$$\begin{aligned} \dot{\xi} &= A_0\xi + F_0\eta + D_\xi w_1 + \sigma_1 \\ \dot{\eta} &= \hat{A}\xi + \hat{B}\eta + D_\eta w_2 \\ \dot{\sigma}_1 &= \sigma_2 \qquad (13) \\ \vdots &= \vdots \\ \dot{\sigma}_{r-1} &= \sigma_r \qquad (14) \\ \dot{\sigma}_r &= A_\sigma\xi + D_\sigma w_\sigma + u \end{aligned}$$

As it was stated at the beginning of this section, when $\sigma = 0$, matrix $A_d = \begin{bmatrix} A_0 & F_0 \\ \hat{B} & \hat{A} \end{bmatrix}$ is made Hurwitz by a correct choice of parameters $F_0, \hat{A}, \hat{B}$. This is posible due to the controllability and observability properties of (8), and it can be done by a number of methods which could include H_∞, as proposed in [1]. The sliding mode can be classically enforced by a discontinuous control depending on σ, in knowledge of an upper bound of w_σ, as was proven in detail in [14] for a system of relative degree two. There remain two unsolved problems with this approach:

One is the fact that global convergence to a neighborhood of the origin of the complete system (8) cannot be assured, it is only possible to guarantee local convergence, i.e. with small enough initial conditions of the unmeasurable state ξ.

The other shortcoming of the mentioned methodology is that, eventhough the assumptions on the Lipschitz property of the derivatives of the disturbance are imposed, in order to select gains of a discontinuous controller, one should have available a known upper bound of this disturbances and their derivatives. Throughout the sliding mode literature it is common to assume such bounds are known, for the perturbation signals and, in some special cases, for their first derivative(s) but not further. In the following section we will present a way of designing the control signal in order to globally draw the states of the complete system, with arbitrary relative degree, to a neighborhood of the origin, without requiring a known upper bound of the complete disturbance vector. The gains of this controller will be tuned, in part, using the available knowledge on the bounds of the perturbations, if any (Assumption f)), and will guarantee that their derivatives will not destroy the system's ultimate boundedness as long as they satisfy Assumptions d and e).

V. MAIN RESULT

The main contribution of this section is to propose a new sliding variable and a control law u_s that enforces the sliding mode, which globally draws the states of a system (8), with arbitrary relative degree, and matched and unmatched disturbances, to a neighborhood of the origin. This control law is composed of a combination of a conventional sliding mode controller, and a linear term that aids to the global convergence. The proposed controller takes advantage of the known bounds of the disturbances, but also ensures that the states of the system will not escape to infinity as long as the disturbances are finite, even with unknown bounds.

By simple inspection of the augmented system (13), it can be noted that it could be viewed as a feedback interconnection of two subsystems with states: $[\xi \quad \eta]^T$ and σ, with σ_1 and ξ as their respective feedback inputs. The results presented in Section II-B will then come in handy, since the satisfaction of the small gain theorem for (13) will guarantee the GAS behaviour of the nominal part of (8), and its ISS properties in presence of disturbances.

Define an array of scalars $K_\sigma = \begin{bmatrix} k_1 & \dots & k_{r-1} \end{bmatrix}$, whose constant value will be defined later, and construct, using the chain of integrators (11) the new sliding variable

$$\zeta(\sigma) = \sigma_r + k_{r-1}\sigma_{r-1} + \cdots + k_1\sigma_1.$$

This new sliding variable has relative degree one, and the closed loop with (11) is

$$\begin{aligned} \dot{\sigma}_\tau =& (A_\tau - B_\tau K_\sigma)\sigma_\tau + B_\tau \zeta \\ \dot{\zeta} =& u_s + A_\sigma \xi + D_\sigma w_\sigma, \end{aligned} \tag{15}$$

where $\sigma_\tau = \begin{bmatrix} \sigma_1 & \dots & \sigma_{r-1} \end{bmatrix}^T$ represents a truncated vector of σ, matrix $A_\tau \in \mathbb{R}^{(r-1\times r-1)}$ is an upper diagonal matrix whose every nonzero element is equal to one, and $B_\tau = \begin{bmatrix} 0 & \dots & 1 \end{bmatrix}^T$. The elements k_i, $i = \{1, \dots, r-1\}$ must be chosen in order to render the matrix formed by $(A_\tau - B_\tau K_\sigma)$ Hurwitz, which is always possible because A_τ and B_τ are in controllability canonical form.

For (15) consider the positive definite and radially unbounded Lyapunov function candidate

$$V_2(\sigma) = \sigma_\tau^T P_2 \sigma_\tau + \frac{1}{2}\zeta^2(\sigma), \tag{16}$$

which satisfies

$$\frac{1}{2}\min\{k_i^2, 1\}|\sigma|^2 \leq V_2(\sigma).$$

Note that $\zeta^2(\sigma) = \sigma^T M_{K_\sigma} \sigma$, where

$$M_{K_\sigma} = \begin{bmatrix} k_1^2 & k_1k_2 & \dots & k_1 \\ & k_2^2 & \dots & k_2 \\ & \star & \ddots & \vdots \\ & & & 1 \end{bmatrix}.$$

In (16), $P_2 = P_2^T > 0$, and it satisfies

$$P_2(A_\tau - B_\tau K_\sigma)^T + (A_\tau - B_\tau K_\sigma)P_2 = -Q_2, \qquad Q_2 > 0.$$

The derivative of (16) over the trajectories of (15), with a control signal as

$$u_s = -K_{lin}\zeta - K_{dis}\,\mathrm{sign}(\zeta)$$

is

$$\begin{aligned} \dot{V}_2 =& -\sigma_\tau^T Q_2 \sigma_\tau + 2\sigma_\tau P_2 B_\tau \zeta - aK_{lin}\zeta^2 - aK_{dis}\zeta\,\mathrm{sign}(\zeta) + \\ & + a\zeta D_\sigma w_\sigma + a\zeta A_\sigma \xi. \end{aligned}$$

Using Young's inequality for the crossed terms in σ_τ, ζ, w_σ and ξ, and choosing K_{dis} such that the following inequality holds:

$$K_{dis} > |D_\sigma|\bar{w}_{\sigma k},$$

leads to

$$\dot{V}_2 \leq -\begin{bmatrix} \sigma_\tau \\ \zeta \end{bmatrix}^T Q_t \begin{bmatrix} \sigma_\tau \\ \zeta \end{bmatrix} + \frac{|A_\sigma|}{2}|\xi|^2 + \frac{|D_\sigma|}{2}|w_\sigma|^2,$$

where $Q_t = \begin{bmatrix} Q_2 & -P_2B_\tau \\ -(P_2B_\tau)^T & K_{lin} - \frac{|A_\sigma|}{2} - \frac{|D_\sigma|}{2} \end{bmatrix}$. Evidently, K_{lin} must be chosen such that $Q_t > 0$.

Function V_2 can be written in the quadratic form

$$V_2 = -\begin{bmatrix} \sigma_\tau \\ \zeta \end{bmatrix}^T P_t \begin{bmatrix} \sigma_\tau \\ \zeta \end{bmatrix},$$

with $P_t = \begin{bmatrix} P_2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

The characteristic values of the pencil $Pe_2 = (Q_t - \lambda P_t)$ have a minimmum defined as

$$\lambda_{\min}(Pe_2) := \min\{\lambda : \det(Pe_2) = 0\}$$

and the following holds [9]

$$\lambda_{min}(Pe_2)V_2 \leq \begin{bmatrix} \sigma_\tau \\ \zeta \end{bmatrix}^T Q_t \begin{bmatrix} \sigma_\tau \\ \zeta \end{bmatrix}.$$

The derivative of V_2 can be bounded as

$$\dot{V}_2 \leq -\lambda_{\min}(Pe_2)V_2 + \frac{|A_\sigma|}{2}|\xi|^2 + \frac{|D_\sigma|}{2}|w_\sigma|^2$$

For system (10) we will consider a quadratic Lyapunov function

$$V_1 = \begin{bmatrix} \xi \\ \eta \end{bmatrix}^T P_1 \begin{bmatrix} \xi \\ \eta \end{bmatrix},$$

where P_1 satisfies

$$P_1 A_d^T + A_d P_1 = -Q_1, \qquad Q_1 > 0.$$

Now we can define the following five $\mathscr{K}_\infty$ functions

$$\begin{aligned} a_1(r) =& \lambda_{min}(Pe_1)r, & \theta_1(r) =& \frac{|P_1|}{\min\{k_i^2, 1\}}r, \\ a_2(r) =& \lambda_{min}(Pe_2)\,r, & \theta_2(r) =& \frac{|A_\sigma|}{2\,\lambda_{min}(P_1)}r, \\ & & \theta_2^w(r) =& \frac{|D_\sigma|}{2}|r|^2, \end{aligned}$$

where $Pe_1 = ((Q_1 - \alpha_1 |P_1| I_2) - \lambda P_1)$ for any $\alpha_1 > 0$, to complete the pair of ISS-Lyapunov functions

$$\dot{V}_1 \leq -a_1(V_1) + \theta_1(V_2),$$
$$\dot{V}_2 \leq -a_2(V_2) + \theta_2(V_1) + \theta_2^w(w_{\sigma b}),$$

which show that both subsystems are ISS with respect to the feedback input, but also that subsystem (15) is ISS with respect to the part of the perturbation with unknown upper bound, and its ISS gain function can be calculated as

$$\gamma_2(r) > a_2^{-1} \circ \theta_2^w(r).$$

From Corollary 1 we can conclude that in order to achieve global stability and convergence to a neighborhood of the origin of (7), gain K_{lin} must be chosen large enough such that the following inequality is satisfied

$$\lambda_{min}(Pe_2) > \frac{|P_1|\,|A_\sigma|}{2\min\{k_i^2, 1\}\,\lambda_{min}(Pe_1)\,\lambda_{min}(P_1)}.$$

This completes a proof for the following theorem:

Theorem 2: The trajectories of a perturbed system in the form (7) can be taken globally and asymptotically to a neighborhood of the origin with a control law

$$u = u_n + u_s \tag{17}$$
$$u_n = -\Gamma_\sigma z - B_\sigma \eta$$
$$u_s = -K_{lin}\zeta - K_{dis}\,\mathrm{sign}(\zeta)$$

with a choice of gains that make the following inequalities hold

$$K_{dis} > D_\sigma |\bar{w}_{\sigma k}| \tag{18}$$
$$Q_t > 0$$
$$\lambda_{min}(Pe_2) > \frac{|P_1|\,|A_\sigma|}{2\min\{k_i^2, 1\}\,\lambda_{min}(Pe_1)\,\lambda_{min}(P_1)}. \tag{19}$$

The trajectories of the system will remain in a neighborhood of the origin bounded by an ISS gain function of the disturbances which can be calculated as a function γ_2 that satisfies

$$\gamma_2(r) > \frac{|D_\sigma|}{2\lambda_{min(Pe_2)}}|r|^2.$$

VI. NUMERICAL EXAMPLE

Consider the unstable plant

$$\dot{x} = \begin{bmatrix} -7 & 3 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & -5 \end{bmatrix} x + \begin{bmatrix} w_1 \\ 0 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix}$$

which is an academic example of a form (7).

Selecting an H_∞ virtual control, the following variables are defined:

$$F_0 = -0.4263,\ \hat{A} = 0.816,\ \hat{B} = -10.044.$$

Gain k_1 is defined as $k_1 = 5$.

The considered disturbances are

$$w_1 = 0.5 + 0.5\sin(5t), \text{ and } w_2 = 1 + 0.4\sin(2t),$$

with bounds $\bar{w_1} = 1.2$, $\bar{w_2} = 1.5$.

Lyapunov function V_1 is defined by $Q_1 = I_2$, and

$$P_1 = \begin{bmatrix} 0.0715 & -0.0003 \\ -0.0003 & 0.0498 \end{bmatrix},$$

and Lyapunov function V_2 by $P_2 = \frac{1}{2}$.

The parameters that allow to find the gains that satisfy the conditions of Theorem 2 are

$\lvert P_1\rvert$	$=$	0.0715	$\lambda_{min}(P_1)$	$=$	0.0498
α_1	$=$	0.5	$\lambda_{min}(Pe_1)$	$=$	13.4880
$\lvert A_\sigma\rvert$	$=$	5	$\min\{k_i^2, 1\}$	$=$	1

which places the condition for the linear gain as $\lambda_{min}(Pe_2) > 0.2663$. For the nonlinear gain, it is assumed that only the bound of the matched disturbance is known, which yields $K_{dis} > 1.5$. For the simulation results the gains were chosen as $K_{dis} = 3$ and $K_{lin} = 10$, which yields $\lambda_{min}(Pe_2) = 1.7379$, and the ISS gain function with respect to the disturbances can be calculated as $\gamma_2(|w|) > 0.2877|w|^2$, wich, with the disturbances considered for this example, leads to the trajectories of the closed loop remaining in a neighborhood of the origin bounded by a number $\varepsilon > 1.0386$. The following simulation results were obtained for the large initial conditions $x_1(0) = 450$, $x_2 = 330$, and $x_3 = 680$.

Figure 1 shows the trajectories followed by the states of the plant, with the gains that satisfy the contidions of Theorem (2). It can be appreciated how the states converge to a neighborhood of the origin and remain below the bound ε. Figure 2 shows the sliging variable that converges to zero, and Figure 3 shows the control signal.

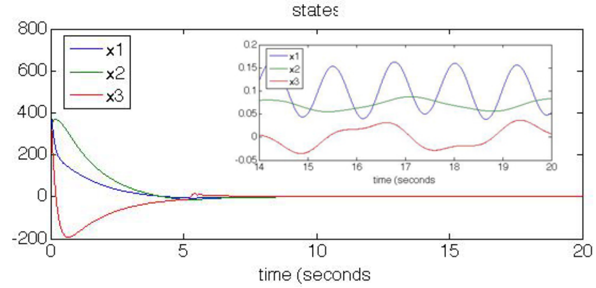

Fig. 1. State trajectories for gains $K_{dis} = 3$ and $K_{lin} = 10$

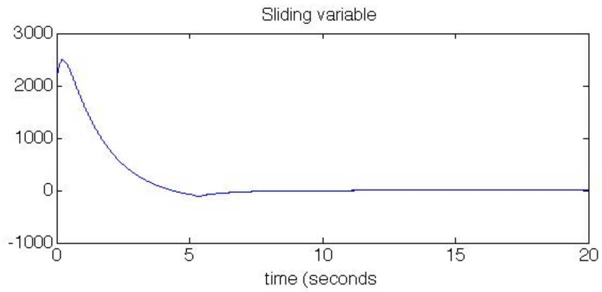

Fig. 2. Sliding variable for gains $K_{dis} = 3$ and $K_{lin} = 10$

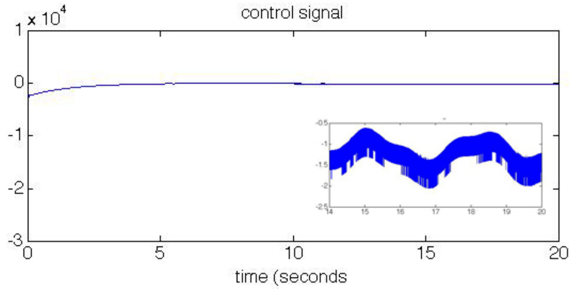

Fig. 3. Control signal for gains $K_{dis} = 3$ and $K_{lin} = 10$

If the gain K_{lin} is selected such that that it does not satisfy the conditions in Theorem 2, for example $K_{lin} = 2$, the simulation results show that the states diverge. This is illustrated in Figure 4.

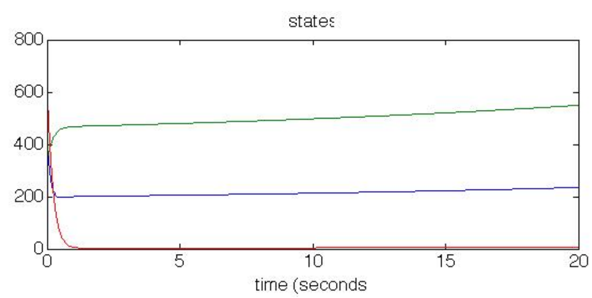

Fig. 4. State trajectories for gains $K_{dis} = 3$ and $K_{lin} = 2$

VII. CONCLUSIONS

In this paper we have addressed the problem of a linear system with matched and unmatched disturbance, arbitrary relative degree, and of which only part of the state is measurable. We have achieved an ISS characterization of such system in a closed loop with the designed controller, when it is subjected to some disturbances of which we know an upper bound, and some of which we don't. This was done by defining a relative degree one sliding surface that allows to reach a virtual control specifically designed to attenuate the unmatched disturbance. A control signal was designed, that enforces the sliding mode and also asures the global ultimate boundedness of the closed loop. Sufficient gains for this controller were found using a pair of ISS Lyapunov functions and a small gain theorem, and simple straight forward conditions for their selection have been given. Although this conditions only provide sufficient gains, it has been proved by simulation that they are not very far from the necessary ones.

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