

# Sliding Motions on $SO(3)$ , Sliding Subgroups

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Abstract—We propose a sliding surface for systems on the Lie group  $SO(3) \times \mathbb{R}^3$ . The sliding surface is shown to be a Lie subgroup. The reduced-order dynamics along the sliding subgroup have an almost globally asymptotically stable equilibrium. The sliding surface is used to design a sliding-mode controller for the attitude control of rigid bodies. The closed-loop system is robust against matched disturbances and does not exhibit the undesired unwinding phenomenon.

## I. INTRODUCTION

The robustness of sliding-mode controllers is a well-known feature that has been extensively documented in the literature. For systems with Euclidean state spaces, the main design principle is well understood: constrain the system trajectories to a, usually, linear sliding surface. Along the sliding surface, the equations of motion are of reduced order and, more importantly, completely independent of matched disturbances. By the linear nature of the sliding surface, the state space of the reduced-order system inherits the Euclidean nature of the full-order system.

However, there are systems which do not evolve naturally on Euclidean spaces. The state space may not be linear, but it may possess other important algebraic structures. Such is the case, e.g., of state spaces which are Lie groups, that is, state spaces which are smooth and satisfy the group axioms for a given smooth product. An interesting example is  $SO(3)$ , which is the natural configuration space for the attitude of rigid bodies. In such a case it seems more intuitive to design a sliding surface which retains the Lie group structure of the full-order system, in other words, a sliding surface which is a Lie subgroup (a sliding subgroup), rather than a linear subspace. In this paper, we propose such a sliding surface for the control of the attitude of rigid bodies.

The rigid body idealization has a considerable amount of applications: robots, spacecraft, aircraft, satellites and underwater vehicles can be modeled as rigid bodies [1], [2]. Problems such as attitude determination, control and estimation can be solved using rigid body models [3], [1].

The motion of a rigid body is represented via Euler equations of motion and kinematic equations which depend on the attitude parametrization [2], [4]. The attitude of a rigid body can be defined using different

representations such as rotation matrices, Euler angles, Rodrigues parameters, modified Rodrigues parameters, quaternions, axis-angle, and many others. Surveys of the attitude representations are given in [4] and [5], where the relations between attitude parametrizations and kinematic equations are detailed.

A natural, unique, and global way to describe the rigid-body attitude is by means of rotation matrices. These matrices form a matrix group called Special Orthogonal, which in the case of 3-D motions is denoted as  $SO(3)$ . Many researchers have studied the control problem in  $SO(3)$ . Within the literature, we can mention [6], where properties like controllability and observability of systems defined on spheres are treated. A proportional-derivative control scheme for systems defined on  $SO(3)$  and  $SE(3)$  is given in [7], and solutions to the output regulation problem for systems defined on matrix Lie groups are presented in [8], [9].

One peculiarity of the control problem in  $SO(3)$  is the impossibility to achieve global stability results using continuous time-invariant feedback, due to the topology of  $SO(3)$  [10], [11], [12]. This problem calls for the notion of almost-global stability, defined in [10] and which is typically needed in systems with rotational motion. Within the works which emphasize the almost-global stability property, we find [12], [8], [7], [13], [14], [15]. The attitude control problem for different systems is considered and almost-global stability is achieved.

## A. Contributions

We propose a sliding surface parametrized with rotational matrices. This automatically ensures that the closed-loop system is globally well defined and that it does not exhibit the so-called unwinding phenomenon. We show that the sliding surface is a Lie group, so it inherits the topological and algebraic properties of the system state space. We show that the controller achieves almost global asymptotic stability, being the global asymptotic stability impossible in the light of the non-contractiveness of  $SO(3)$ .

## B. Paper Structure

In Section II we consider the attitude control problem about a single axis of rotation. We show that, if we insist on a linear surface in a Euclidean coordinate chart, the sliding surface will necessarily break into disjoint subsets. Section III contains all the preliminary material, mostly about  $SO(3)$  and the dynamics of a rotating rigid body.

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The controller is proposed and the stability of the closed loop system is analyzed in Section IV. The performance of the controller is further assessed with simulations in Section V. Conclusions are given in the last section.

## II. MOTIVATIONAL EXAMPLE, SLIDING MOTIONS ON $S \times \mathbb{R}$

To illustrate the potential problems of linear sliding surfaces on systems with non-Euclidean configuration spaces we elaborate on an example given in [11]. Although the scope of [11] is restricted to smooth control laws, the essence of the problems remains in the non-smooth setting.

Consider a rigid body with a single fixed axis of rotation. Its phase space  $\mathcal{M}$  consists of all possible rotations,  $S$ , together with all possible angular velocities,  $\mathbb{R}$ , so that  $\mathcal{M}$  is the cylinder  $S \times \mathbb{R}$ . Suppose further that the system is subject to a control torque,  $u$ . In local coordinates  $(\theta, \omega)$  for  $\mathcal{M}$ , the equations of motion are

$$\dot{\theta} = \omega \quad (1a)$$

$$\dot{\omega} = u. \quad (1b)$$

The objective is to regulate the angular position to the value  $\theta = 0$ . A typical linear sliding variable could be

$$\sigma(\theta, \omega) = \omega + \theta. \quad (2)$$

It is straightforward to verify that the control law

$$u(\theta, \omega) = -(|\omega| + 1) \text{sign} \sigma(\theta, \omega) \quad (3)$$

drives the state to the sliding surface

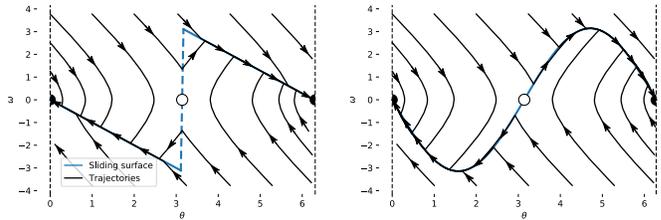
$$\mathcal{D} = \{(\theta, \omega) \in \mathbb{R} \times \mathbb{R} \mid \sigma(\theta, \omega) = 0\}$$

(solutions are taken in the sense of Filippov [16]). Also, it is easily seen that the reduced-order dynamics along the sliding surface are given by  $\dot{\theta} = -\theta$ , for which  $\theta = 0$  is globally exponentially stable. It may thus appear that the closed-loop system (1)-(3) has a globally exponential stable equilibrium. However, note that a given point is not represented uniquely by  $\theta$ . Namely, given a principal angle  $\bar{\theta} \in [0, 2\pi)$ , all  $\theta$  such that  $\theta \bmod 2\pi = \bar{\theta}$  represent the same point in  $S$ . Moreover, some of these  $\theta$  yield different control values. Thus, the control is, in reality, multi-valued, a fact that has been ignored in the previous analysis and which invalidates the hasty conclusion about the global asymptotic stability of  $(0, 0)$ .

A direct practical consequence of the non-unique representation is the unwinding phenomenon. To appreciate it, consider an initial condition  $(4\pi, 0) \notin \mathcal{D}$ . Although this initial condition represents precisely the desired equilibrium  $(0, 0) \in \mathcal{D}$ , the control law (3) will drive the system state from  $(4\pi, 0)$  to  $(0, 0)$  by making two absolutely unnecessary rotations.

We can stop the unwinding phenomenon if we replace (2) with

$$\sigma(\theta, \omega) = \omega - \pi + (\theta - \pi) \bmod 2\pi, \quad (4)$$



(a) Sliding variable (4). The sliding surface is disconnected and the vector field is discontinuous at some point outside it. (b) Sliding variable (5). The sliding surface is connected and smooth.

Fig. 1: Phase planes of (1), (3). We identify pairs of points  $(0, \omega)$  and  $(2\pi, \omega)$ , so that the phase planes lie on  $S \times \mathbb{R}$ . There are a stable and an unstable equilibrium at  $(0, 0)$  and  $(0, \pi)$ , respectively.

as this makes the control single-valued. Note that the sliding variable is still linear for  $\theta \in [0, 2\pi] \setminus \{\pi\}$  but, because of the topology of the cylinder, it must necessarily contain a discontinuity at some point in order to retain its linear character. In this case, we have chosen to place such discontinuity at  $\theta = \pi$ .

Figure 1a shows the phase plane of (1), (3), (4). For this simple two-dimensional example, it suffices to look at the phase plane to see that the system enjoys almost global asymptotic stability (see Definition 1 on p. 4), which is the most we can expect from a mechanical system with a compact configuration space. However, note that  $\mathcal{D}$  is not connected, and that the vector field has discontinuities outside  $\mathcal{D}$ , along the dashed line joining the points  $(\pi, -\pi)$  and  $(\pi, \pi)$ . Also, note that there is an unstable equilibrium at  $(\pi, 0)$  and that its presence rules out the global attractivity to  $(0, 0)$ . A rigorous analysis of the global stability properties of this system requires the use of mathematical tools not commonly found in the sliding-mode literature. Indeed, with few exceptions such as [17],  $\sigma$  is always required to be continuous.

A reasonable alternative is to replace (2) by

$$\sigma(\theta, \omega) = \omega + \pi \sin \theta. \quad (5)$$

The periodicity in  $\theta$  ensures that (3) is single-valued without destroying the continuity of the sliding variable. The phase plane is shown in Figure 1b. Again,  $(0, 0)$  is almost globally asymptotically stable, but without the technical difficulties mentioned above.

In Section IV we propose a smooth sliding variable for systems whose configuration space is  $SO(3)$ . The sliding variable is such that, in the same spirit of (5), yields a single-valued controller that exhibits no unwinding. The stability properties of the system will be established rigorously (see Remark 1).

## III. PRELIMINARIES, SYSTEMS ON $SO(3) \times \mathbb{R}^3$

In this section, we derive the error equations for the attitude of a rigid body. We also recall some stability

notions.

### A. $SO(3)$

Since the set of possible attitudes of a rigid body is not a Euclidean space, several parametrizations are used in the literature. Common parametrizations include Euler angles, quaternions, and rotational matrices. Parametrizations using Euler angles contain singular points, that is, points at which the Jacobian of the coordinate chart loses rank and angular velocities are no longer well defined. Also, Euler angles are non-unique, which leads to the unwinding phenomenon described in the previous section. Parametrizations using quaternions are global, by which we mean that there are no singular points. However, quaternion representations are still non-unique (as every attitude is represented by two quaternions) and the unwinding problem can still potentially manifest.

The set of rotation matrices forms the Special Orthogonal group

$$SO(3) = \{R \in \mathbb{R}^{3 \times 3} \mid RR^\top = I, \det(R) = 1\}$$

with  $I$  the identity matrix of order 3. Rotational matrices parametrize the set of attitudes, inasmuch as every attitude is associated with a single rotational matrix and vice versa. Parametrizations using rotational matrices have the compelling advantage of being both global and unique. The interested reader is referred to [12], where a more thorough analysis of these different parametrizations can be found.

Besides being a smooth manifold,  $SO(3)$  has the structure of a Lie group with the standard matrix product. The tangent space at the identity of  $SO(3)$ , denoted  $T_I SO(3)$ , can be associated with the space of skew-symmetric matrices  $X \in \mathbb{R}^{3 \times 3}$ ,  $X + X^\top = 0$ . Endowing this space with the bracket  $[A, B] = AB - BA$  gives rise to the Lie algebra of  $SO(3)$ , denoted by  $\mathfrak{so}(3)$  (see [2, Appx. A] for details and formal definitions).

Let  $\times$  be the standard cross product in  $\mathbb{R}^3$ . We will make use of the isomorphism  $\omega \mapsto \omega^\times$  between  $(\mathbb{R}^3, \times)$  and  $(\mathfrak{so}(3), [\cdot, \cdot])$ , defined as

$$\omega^\times = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

For every  $(R, \omega)$ , we have [18]

$$(R\omega)^\times = R\omega^\times R^\top. \quad (6)$$

We will also make use of the inverse map of  $(\cdot)^\times$ , denoted by  $\text{vex} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$ .

The operator  $\mathbb{P}_a : \mathbb{R}^{3 \times 3} \rightarrow \mathfrak{so}(3)$  retains the skew symmetric part of a matrix,

$$\mathbb{P}_a(A) = \frac{1}{2}(A - A^\top),$$

while  $\mathbb{P}_s(A) : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$ ,

$$\mathbb{P}_s(A) = \frac{1}{2}(A + A^\top),$$

retains the symmetric part.

The trace will serve as a matrix inner product on  $\mathbb{R}^{3 \times 3}$ ,

$$\langle A, B \rangle = \text{tr}(A^\top B).$$

If  $v, w \in \mathbb{R}^3$  and  $A = A^\top$ , we know that [18]

$$\langle v, w \rangle = \frac{1}{2} \langle v^\times, w^\times \rangle \quad (7)$$

and

$$\langle A, v^\times \rangle = 0. \quad (8)$$

Recall that, by Euler's rotation theorem, any attitude  $R$  can be attained by a single rotation  $\theta \in [0, 2\pi)$  about some unit vector  $\eta \in \mathbb{R}^3$ . Rodrigues formula establishes that

$$R = I + \eta^\times \sin(\theta) + (\eta^\times)^2 (1 - \cos(\theta)). \quad (9)$$

It follows from Rodrigues formula that [2, Ch. 2]

$$\text{tr}(R) = 1 + 2 \cos(\theta). \quad (10)$$

### B. Attitude Dynamics

The state space  $\mathcal{M}$  of the attitude dynamics consists of the set of possible attitudes,  $SO(3)$ , together with the set of possible angular velocities,  $\mathbb{R}^3$ , so that  $\mathcal{M} = SO(3) \times \mathbb{R}^3$ .

Suppose we are able to provide torques along the principal axes of a rigid body with an inertia matrix  $J \in \mathbb{R}^{3 \times 3}$ ,  $J = J^\top > 0$ . Then, the rotational motion of the rigid body is determined by a kinematic equation

$$\dot{R} = R\omega^\times$$

(for conciseness we omit all time arguments), and Euler's equation,

$$J\dot{\omega} = (J\omega) \times \omega + u + d,$$

where  $u, d \in \mathbb{R}^3$  are the control and the disturbance torques, respectively [2, Ch. 2].

Let  $R_d$  and  $\omega_d$  be (possibly time varying) desired attitude and angular velocity satisfying the kinematic equation

$$\dot{R}_d = R_d \omega_d^\times. \quad (11)$$

Now, define the attitude and angular velocity errors  $R_e = R_d^\top R$  and  $\omega_e = \omega - R_e^\top \omega_d$ , respectively, and note that

$$(R_e, \omega_e) = (I, 0) \quad \text{if, and only if,} \quad (R, \omega) = (R_d, \omega_d).$$

The error kinematics are then

$$\begin{aligned} \dot{R}_e &= (R_d \omega_d^\times)^\top R + R_d^\top R \omega^\times \\ &= -(\omega_d)^\times R_e + R_e \omega^\times \\ &= (R_e (\omega_e - \omega))^\times R_e + R_e \omega^\times. \end{aligned}$$

Using (6) we obtain

$$\dot{R}_e = R_e \omega_e^\times. \quad (12a)$$

The dynamic equation is

$$\begin{aligned} J\dot{\omega}_e &= J(\dot{\omega} - \dot{R}_e^\top \omega_d - R_e^\top \dot{\omega}_d) \\ &= (J\omega) \times \omega + J\omega_e^\times R_e^\top \omega_d - JR_e^\top \dot{\omega}_d + u + d \\ &= (J\omega) \times \omega + JR_e^\top ((R_e \omega_e)^\times \omega_d - \dot{\omega}_d) + u + d, \end{aligned}$$

where we have used (6) to obtain the last equality. Set

$$u = -JR_e^\top ((R_e \omega_e)^\times \omega_d - \dot{\omega}_d) + v$$

so that

$$J\dot{\omega}_e = (J\omega) \times \omega + v + d, \quad (12b)$$

### C. Stability

It follows from the compactness of  $SO(3)$  and the arguments presented in [11] that global asymptotic stability of an equilibrium of (12) is impossible, so we must settle for almost global asymptotic stability, a property that we recall now.

A dynamical system is almost globally asymptotically stable if all trajectories starting in some open dense subset of the state space tend asymptotically to a specified stable equilibrium state.

Almost global stability will be proved using a refinement of the LaSalle invariance principle for systems defined on manifolds.

[LaSalle principle [19]] For a smooth vector field  $f$  on  $\mathcal{M}$ , let  $\mathcal{A} \subset \mathcal{M}$  be compact and positively invariant for  $f$ . Let a smooth function  $V : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\dot{V}(x) \leq 0$  for all  $x \in \mathcal{A}$  and let  $\mathcal{B}$  be the largest positively invariant set for  $f$  contained in  $\{x \in \mathcal{A} \mid \dot{V}(x) = 0\}$ . Then, the following statements hold:

- Each integral curve of  $f$  with initial condition in  $\mathcal{A}$  approaches  $\mathcal{B}$  as  $t \rightarrow +\infty$ .
- If  $\mathcal{B}$  consists of a finite number of isolated points, then each integral curve of  $f$  with initial condition in  $\mathcal{A}$  converge to a point of  $\mathcal{B}$  as  $t \rightarrow -\infty$ .

## IV. MAIN RESULTS,

### SLIDING-MODE CONTROL ON $SO(3) \times \mathbb{R}^3$

Allow us first to endow  $\mathcal{M} = SO(3) \times \mathbb{R}^3$  with the product  $\mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$  defined by

$$(R_1, \omega_1) \cdot (R_2, \omega_2) = (R_3, \omega_3), \quad (13a)$$

where

$$R_3 = R_1 R_2 \quad (13b)$$

$$\omega_3^\times = \mathbb{P}_a \left( R_1 \omega_2^\times + R_2^\top \omega_1^\times - \frac{1}{2} [R_1, R_2^\top] \right). \quad (13c)$$

$\mathcal{M}$  is a Lie group with the product (13).

Proof:  $\mathcal{M}$  is a smooth manifold and (13) is clearly smooth, so we will only verify the group axioms.

The point  $(I, 0)$  is the identity:

$$(I, 0) \cdot (R, \omega) = (R, \text{vex}(\mathbb{P}_a(\omega^\times I + 0))) = (R, \omega).$$

The multiplicative inverse of  $(R, \omega)$  is  $(R^\top, -\omega)$ :

$$\begin{aligned} (R^\top, -\omega) \cdot (R, \omega) &= (I, \text{vex}(\mathbb{P}_a(R^\top \omega^\times - R^\top \omega^\times + 0))) \\ &= (I, 0). \end{aligned}$$

### A. The Proposed Sliding Subgroup

The sliding surface is introduced in the following proposition. Consider the sliding variable  $\sigma : \mathcal{M} \rightarrow \mathbb{R}^3$  defined by

$$\sigma(R_e, \omega_e) = \omega_e + \text{vex}(\mathbb{P}_a(R_e)). \quad (14)$$

The sliding surface

$$\mathcal{D} = \{(R_e, \omega_e) \in \mathcal{M} \mid \sigma(R_e, \omega_e) = 0\} \quad (15)$$

is a Lie subgroup of  $\mathcal{M}$ .

Proof: The smoothness of  $\mathcal{D}$  is a simple consequence of the smoothness of  $\text{vex}$  and  $\mathbb{P}_a$ . We now verify the group axioms.

The identity is in  $\mathcal{D}$ :  $\sigma(I, 0) = 0 + \text{vex}(\mathbb{P}_a(I)) = 0$ .

$\mathcal{D}$  is closed under multiplication: The product (13) gives

$$\begin{aligned} \sigma((R_1, \omega_1) \cdot (R_2, \omega_2))^\times &= \\ \mathbb{P}_a \left( R_1 \omega_2^\times + R_2^\top \omega_1^\times - \frac{1}{2} [R_1, R_2^\top] \right) &+ \mathbb{P}_a(R_1 R_2). \end{aligned}$$

Now, let  $(R_1, \omega_1)$  and  $(R_2, \omega_2)$  be both in  $\mathcal{D}$ . We have

$$\begin{aligned} \sigma((R_1, \omega_1) \cdot (R_2, \omega_2))^\times &= \\ \mathbb{P}_a \left( R_1 R_2 - \frac{1}{2} [R_1, R_2^\top] - R_1 \mathbb{P}_a(R_2) - R_2^\top \mathbb{P}_a(R_1) \right). \end{aligned}$$

Since

$$\begin{aligned} R_1 R_2 - \frac{1}{2} [R_1, R_2^\top] - R_1 \mathbb{P}_a(R_2) - R_2^\top \mathbb{P}_a(R_1) &= \\ R_1 R_2 + R_2^\top R_1^\top & \end{aligned}$$

and the right-hand side is symmetric, we have

$$\sigma((R_1, \omega_1) \cdot (R_2, \omega_2)) = 0,$$

which shows that  $(R_1, \omega_1) \cdot (R_2, \omega_2)$  is also in  $\mathcal{D}$ .

$\mathcal{D}$  is closed under inversion: Suppose that  $(R, \omega) \in \mathcal{D}$ . We will verify that  $(R^\top, -\omega) \in \mathcal{D}$ . Direct computation gives

$$\sigma(R^\top, -\omega)^\times = -\omega^\times + \mathbb{P}_a(R^\top) = -\omega^\times - \mathbb{P}_a(R) = 0. \quad \blacksquare$$

Note that the sliding surface inherits both the topological and algebraic properties of the phase space. This stands in contrast with the standard linear approach.

### B. Stability of the Reduced-Order System

The dynamics along  $\mathcal{D}$  are obtained by simply enforcing the constraint  $\sigma(R_e, \omega_e) = 0$  on (12). These are

$$\dot{R}_e = -R_e \mathbb{P}_a(R_e). \quad (16)$$

The identity  $R_e = I$  is an almost globally stable equilibrium of the reduced-order dynamics (16).

Proof: We borrow the linear Lyapunov-function candidate [18]

$$V_R(R_e) = \frac{1}{2} \text{tr}(I - R_e).$$

Note that  $V_R(R_e) \geq 0$  by the facts that  $\text{tr}(I) = 3$  and that  $\text{tr}(R_e) \in [-1, 3]$  (cf. (10)), and that  $V_R(R_e) = 0$  if, and only if,  $R_e = I$ . Its time derivative is

$$\begin{aligned}\dot{V}_R(R_e) &= \frac{1}{2} \text{tr}(R_e \mathbb{P}_a(R_e)) \\ &= \frac{1}{2} \langle R_e^\top, \mathbb{P}_a(R_e) \rangle \\ &= \frac{1}{2} \langle -\mathbb{P}_a(R_e) + \mathbb{P}_s(R_e), \mathbb{P}_a(R_e) \rangle \\ &= -\frac{1}{2} \langle \mathbb{P}_a(R_e), \mathbb{P}_a(R_e) \rangle,\end{aligned}$$

where we have used (8) to derive the last equation. Using (7) we obtain

$$\begin{aligned}\dot{V}_R(R_e) &= -\langle \text{vex}(\mathbb{P}_a(R_e)), \text{vex}(\mathbb{P}_a(R_e)) \rangle \\ &= -\|\text{vex}(\mathbb{P}_a(R_e))\|^2 \leq 0.\end{aligned}$$

This proves the stability of  $R_e = I$ .

Almost global convergence will be established using Theorem 1. Allow us to first compute the set

$$\mathcal{E} = \left\{ R_e \in SO(3) \mid \dot{V}_1(R_e) = 0 \right\}.$$

It follows from (9) that

$$\mathbb{P}_a(R_e) = \sin(\theta) \eta_e^\times.$$

Since  $\eta_e \neq 0$ , we have  $\mathbb{P}_a(R_e) = 0$  only if  $\theta \in \{0, \pi\}$ . From (9) we reach the conclusion

$$\mathcal{E} = \{I\} \cup \{-I + 2\eta\eta^\top \mid \eta \in \mathbb{R}^3, \|\eta\| = 1\}.$$

Note that  $\mathcal{E}$  is comprised of equilibria, so  $\mathcal{E}$  is in fact the largest invariant set contained in  $\mathcal{E}$ , so all trajectories converge to  $\mathcal{E}$ .

Finally, note that for

$$R_e \in \{-I + 2\eta\eta^\top \mid \eta \in \mathbb{R}^3, \|\eta\| = 1\}$$

we have  $V_R(R_e) = \text{tr}(I - \eta\eta^\top) = 3 - \text{tr}(\eta^\top \eta) = 2$ , which is the maximum value that  $V_R$  can attain. This implies that all the equilibria  $R_e \in \mathcal{E}$  are unstable except for  $R_e = I$ . Thus, the only attractor is the point  $R_e = I$ . The fact that  $\mathcal{E}$  is of measure zero allows us to conclude almost global convergence.  $\blacksquare$

The problem of Section II and the proposed sliding surface can be recovered from the  $SO(3) \times \mathbb{R}^3$  setting by restricting the motion about a single axis.

### C. The Reaching Law

The condition  $\sigma(R_e, \omega_e) = 0$  can be enforced with the following control law.

Suppose that the perturbations  $d$  are bounded by a known constant  $\bar{d} > 0$ , that is,  $\|d\| \leq \bar{d}$ . Take a constant  $\delta > 0$  and set

$$v(R_e, \omega_e, \omega) = -K(\omega_e, \omega) \frac{\sigma(R_e, \omega_e)}{\|\sigma(R_e, \omega_e)\|} \quad (17)$$

with  $\sigma$  as in (14) and with the gain

$$K(\omega_e, \omega) \geq \|J\|_2 \cdot \|\omega\|^2 + \|\omega_e\| + \bar{d} + \delta. \quad (18)$$

Then, the trajectories of (12) converge to  $\mathcal{D}$  (15) in finite time.

Proof: Consider the Lyapunov candidate function

$$V_\sigma(\sigma) = \frac{1}{2} \sigma^\top J \sigma.$$

Its time derivative is, according to (14),

$$\dot{V}_\sigma(\sigma) = \sigma^\top J \left( \dot{\omega}_e + \text{vex}(\mathbb{P}_a(\dot{R}_e)) \right).$$

The dynamics (12) give

$$\begin{aligned}\dot{V}_\sigma(\sigma) &= \sigma^\top J \left( \dot{\omega}_e + \text{vex}(\mathbb{P}_a(\dot{R}_e)) \right) \\ &= \sigma^\top \left( (J\omega) \times \omega + \text{vex}(\mathbb{P}_a(R_e \omega_e^\times)) + d + v \right).\end{aligned}$$

It is not difficult to verify that

$$\|(J\omega) \times \omega\| \leq \|J\|_2 \cdot \|\omega\|^2$$

and that (7) implies that

$$\|\text{vex}(\mathbb{P}_a(R_e \omega_e^\times))\| = \|\omega_e\|.$$

Thus, the time derivative is bounded as

$$\dot{V}_\sigma(\sigma) \leq -\|\sigma\| \left( K(\omega_e, \omega) - \|J\|_2 \cdot \|\omega\|^2 - \|\omega_e\| - \bar{d} \right).$$

Condition (18) ensures that

$$\dot{V}_\sigma(\sigma) \leq -\delta \sqrt{\frac{V_\sigma(\sigma)}{\lambda_{\max}(J)}}.$$

The Comparison Lemma implies that  $V_\sigma$ , and hence  $\sigma$ , go to zero in finite time.  $\blacksquare$

## V. SIMULATIONS

In this section we show the unwinding phenomenon for a controller parametrized with quaternions. We also illustrate the performance of the proposed controller using simulations.

### A. The Unwinding Phenomenon in Quaternion Representations

We compare a sliding-mode controller based on quaternions with our controller to illustrate the potential unwinding problems of the quaternion approach.

Popular attitude representations are based on quaternions. Even though global representations are obtained (there are no singularities), the non-uniqueness of quaternion representations lead to undesired unwinding. In the following example we simulate the behavior of a rigid body with a sliding-mode controller based on quaternions [20] and with the controller (17).

In terms of the unit quaternion  $q = (q_0, q_v)$ ,  $q_0 \in \mathbb{R}$ ,  $q_v \in \mathbb{R}^n$ , the attitude dynamics of a rigid body are given by

$$\begin{aligned}J\dot{\omega} &= (J\omega) \times \omega + u + d \\ \dot{q}_0 &= -\frac{1}{2} q_v^\top \omega \\ \dot{q}_v &= \frac{1}{2} (q_0 I + q_v^\times) \omega\end{aligned} \quad (19)$$

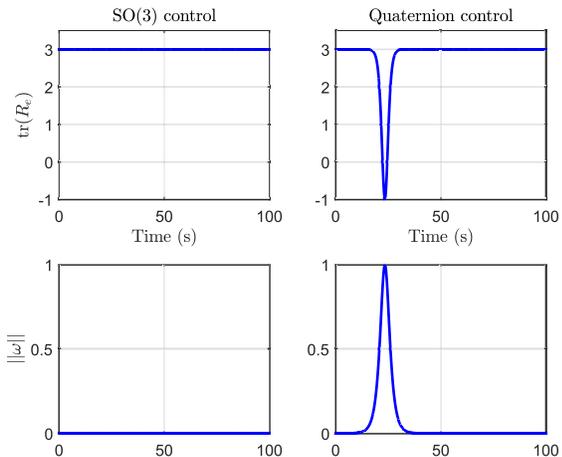


Fig. 2: Comparison between sliding-mode attitude controllers. The controller based on quaternion representations exhibits the unwinding phenomenon, the one based on rotation matrices does not.

Consider the sliding-mode controller with linear sliding variable

$$\begin{aligned}\sigma_q(q_v, \omega) &= q_v + \omega \\ u(q_v, \omega) &= -k_q \frac{\sigma_q(q_v, \omega)}{\|\sigma_q(q_v, \omega)\|}.\end{aligned}\quad (20)$$

As before, the gain  $k_q$  is chosen large enough to enforce the sliding motion (details are not shown).

For concreteness, we take  $J = \text{diag}(3, 4, 5)$  as the inertia matrix. The disturbances are given by

$$d(t) = (\sin(5\pi t) \quad \cos(7\pi t) \quad \sin(9\pi t))^T.$$

The gain (18) is set as

$$K(\omega_e, \omega) = 7 \cdot \|\omega\|^2 + 2 \cdot \|\omega_e\| + 1.8, \quad (21)$$

while we set  $k_q = 5$ .

We consider the regulation problem  $R_d = I$ ,  $\omega_d = 0$ . To illustrate the unwinding phenomenon, suppose that the system starts at the desired attitude,  $R(0) = I$ . In terms of quaternions, the attitude is represented either as

$$q_1 = (-1 \quad 0 \quad 0 \quad 0)^T \quad \text{or} \quad q_2 = (1 \quad 0 \quad 0 \quad 0)^T.$$

Although they both represent the same attitude,  $q_1$  is unstable while  $q_2$  is stable. Allow us to initiate the system on the unstable equilibrium  $q(0) = q_1$ .

Figure 2 shows the trace of  $R_e$  (a measure of how close  $R_e$  is to the identity) and  $\|\omega\|$  for systems (12) and (19), under the action of the controllers (17) and (20), respectively. It can be seen that the control law (17) keeps the attitude at the desired location, i.e., where  $\text{tr}(R_e) = 3$  and  $\|\omega\| = 0$ . On the other hand, even though system (19) starts at the desired attitude, the control (20) drives the state away from the desired attitude and converges again after a full turn.

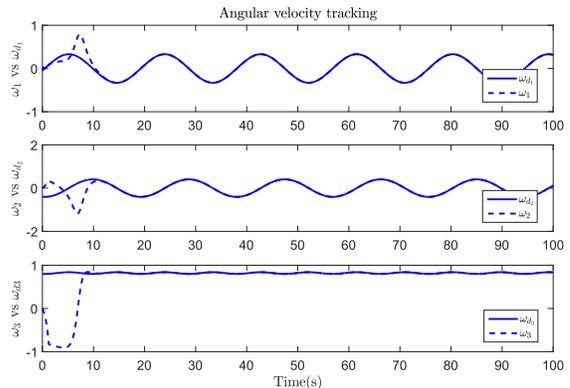


Fig. 3: Desired and actual angular velocities. Sliding-mode controller (17) in  $SO(3)$ .

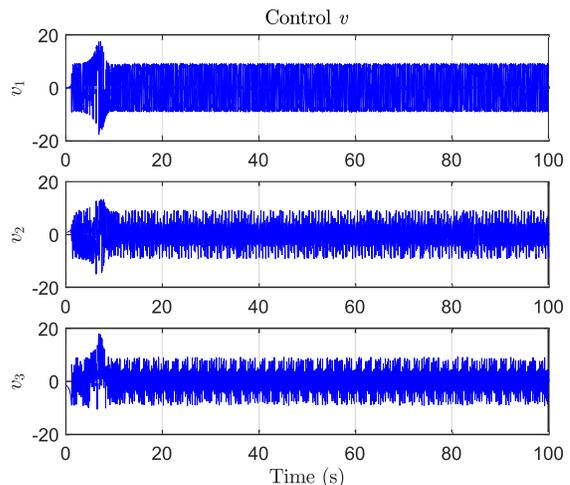


Fig. 4: Control torques  $u$  produced by the control law (17).

## B. Controller Performance for a Tracking Problem

Now we consider a time-varying reference satisfying (11) with the desired angular velocity shown in Figure 3. The figure also shows the actual angular velocity for the system controlled with (17). It can be seen that  $\omega_e \rightarrow 0$ . Figure 4 shows the control torques. The trace of the attitude error matrix is shown in Figure 5. It shows that  $\text{tr}(R_e) \rightarrow 3$  which necessarily means that  $R \rightarrow R_d$ .

## VI. CONCLUSIONS AND FUTURE WORK

If the control objective is simply to specify the local behavior of a system evolving on a manifold, then the choice of parametrization of the manifold is unimportant. If, however, one is interested in specifying the global behavior and the system in question evolves on a non-Euclidean manifold, then such choice becomes crucial. In the specific attitude control problem, quaternions supersede Euler angles in that there are no singularities, but still ail from the non-uniqueness problem. Rotational matrices, on the other hand, are singularity free and unique.

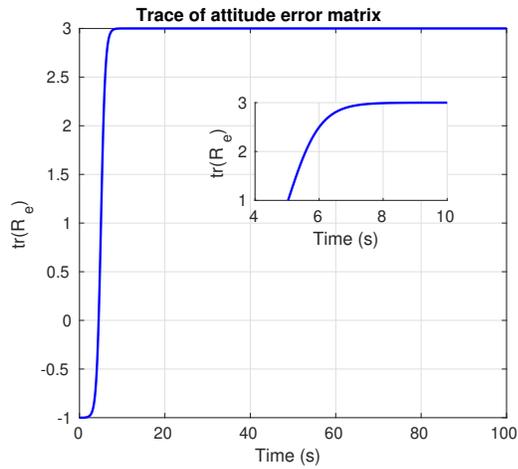


Fig. 5: Trace of attitude error matrix  $R_e$  which shows that the closed-loop system (12), (17) attains the required attitude, i.e.  $R_e = R_d^T R \rightarrow I$ .

We have proposed a sliding surface defined in terms of rotational matrices. The resulting controller is thus singularity free and does not induce the unwinding phenomenon. It is shown that the reduced order system is almost globally asymptotically stable, and that the sliding surface is in fact a Lie subgroup of the state space  $SO(3) \times \mathbb{R}^3$ . The latter property ensures the smoothness of the sliding surface and can be used as a general design principle for systems that live on other Lie groups.

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